PHYS411 Notes (Fall 2022)

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1 Electrostatics

1.1 Electric Fields

We think of charge as a continuous variable (even though in reality we know they are quantized). We begin by considering point-like charges, each with some magnitude of charge. Suppose we have two point charges q and q', with position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. They have some distance between them $\boldsymbol{R} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. The force between these two objects is given by Coulomb's Law:

$$\boldsymbol{F} = \frac{qq'}{4\pi\epsilon_0 R^2}\boldsymbol{R}$$

We measure charge in units of Coulombs, and the force is in units of Newtons. ε_0 is a constant, $\epsilon_0 = 8.85 \times 10^{-12}\frac{C^2}{Nm^2}$.

We claim that a point charge generates a field known as the electric field, which is a function of space, $\boldsymbol{E}(\boldsymbol{r})$. This is a vector-valued field, it outputs a vector for every point in space. This field interacts with any charges placed in the field. If we have a field $\boldsymbol{E}$, the force on a point charge q is $\boldsymbol{F} = q\boldsymbol{E}$.

Suppose we have two charges in space. Each of them produces its own electric field. The superposition principle states that the net electric field at a point in space is the sum of the electric fields:

$$\boldsymbol{E}_{\text{total}} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \cdots = \sum_i \frac{q_i}{4\pi\epsilon_0}\frac{\boldsymbol{R}_i}{R_i^2}$$

Where we have generalized to N point-like charges.

Suppose we have a Cartesian coordinate system, with a charge q_1 at $\boldsymbol{r}_1$. The field generated by this charge will be given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{\boldsymbol{R}}{R^2}$$

Where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_1$.

If we have a distribution of charge on a surface, with some given volume charge density ρ, we can integrate to get the total charge:

$$\int \rho(\boldsymbol{r})d\boldsymbol{r} = Q$$

We can actually represent a system of point like charges using a charge density, using a dirac delta:

$$\rho(\boldsymbol{r}) = \sum_{i=1}^{N} q_i\delta^3(\boldsymbol{r} - \boldsymbol{r}_i)$$

which intuitively is a series of spikes, one at every point charge's location. If we sub this into the definition of the electric field:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\boldsymbol{r'}\,\rho(\boldsymbol{r})\frac{\boldsymbol{r} - \boldsymbol{r'}}{|\boldsymbol{r} - \boldsymbol{r'}|^2}$$

This is technically all we need, but this can be extremely painful when doing problems, so now we need to look at ways to make this easier. To simplify this, let us pull out things we know about vector fields. The first thing that we pull out is the flux of a vector field. The flux is obtained by taking some small differential plane $d\boldsymbol{s}$, where the magnitude is the area of the surface and the direction is the normal vector of the plane. We can then compute the dot product with the electric field.

$$d\Phi = \boldsymbol{E} \cdot d\boldsymbol{s}$$

Suppose we generate a cube, and place it in the field, and we want to find the flux through the cube. Suppose one corner is at position (x, y, z), and the opposite corner is at $(x + dx, y + dy, z + dz)$. We have 6 surfaces to compute the flux through. Suppose we look at the surface that is along the $\boldsymbol{x}$ direction. The area of this surface is $dy\, dz$. This tells us that the flux through that surface is $E_x\, dy\, dz$, and we have that the flux through the opposite surface is also $E_x\, dy\, dz$! However, we have made a mistake here, the fields are at different places, so its not both E_x. We find that the correct flux in that direction through the cube gets $\frac{\partial E_x}{\partial x}\, dx\, dy\, dz$. If we compute the other 4 faces, we will intuitively obtain

$$d\Phi = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dx\, dy\, dz = \nabla \cdot \boldsymbol{E}\, dx\, dy\, dz$$

This is the divergence of the field. This gives us an intuitive idea for what the divergence is. If the total flux through the cube is 0, this means that the field is constant, which means the divergence tells us the source of the field. By definition, what we have computed is the same as

$$d\Phi = \oint_V \boldsymbol{E} \cdot d\boldsymbol{s}$$

We can then generalize this to a larger surface, and we still maintain our volume integral:

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{s} = \int d^3\boldsymbol{r}\, \nabla \cdot \boldsymbol{E} = \frac{Q}{\epsilon_0}$$

This is Gauss's theorem, and is the integral form of Maxwell's first equation.

This is useful in many cases. Suppose we have a sphere of radius R, with total charge Q. We have that $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$. We could just do out the integral using the definition of an $\boldsymbol{E}$ field, but there is an easier way. Suppose we have a sphere of radius r surrounding the sphere, (this is a Gaussian surface). We can argue that the electric field inside the Gaussian sphere is equal in magnitude everywhere. We can also argue that the direction of the electric field must be pointing outwards. We can apply Gauss's theorem, where the surface element is moving in the same direction of $\boldsymbol{E}$. Since $\boldsymbol{E}$ is the same everywhere, we can pull out the E, and integrate over the surface:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{s} = E \oint d\boldsymbol{s} = 4\pi E R^2 = \frac{Q}{\epsilon_0}$$

This has simplified down to a matter of algebra:

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

We see that we have done almost no work, and yet we have the answer that we wanted.

Let us ask a nastier question. What is the electric field at some point inside the sphere? What we do is we create a Gaussian surface inside the sphere, and do the same thing, ignoring the electric field generated by the charge outside of the Gaussian surface (since it cancels out).

How do we calculate the divergence of a vector field? If we have an electric field, $\boldsymbol{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r'})\boldsymbol{R}}{R^2} d^3\boldsymbol{r'}$, and some surface V.

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r})$$

How do we compute this? Let us assume that we can interchange the divergence operation with the integral. If we do this, we end up computing the divergence of $\boldsymbol{R}$. This in turn will turn into a divergence of $\boldsymbol{r}$. This turns out to give us $\nabla \cdot \frac{\boldsymbol{r}}{r^3} = 4\pi\delta^3(\boldsymbol{r})$. This is one of the most important formulas for this course. This leaves us with

$$\nabla \cdot \boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r})\nabla \cdot \boldsymbol{R}}{R^2}\, d^3\boldsymbol{r'} = \int \frac{\rho(\boldsymbol{r})}{4\pi\epsilon_0} 4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r'})\, d^3\boldsymbol{r'} = \int \frac{\rho(\boldsymbol{r})}{\epsilon_0}\delta^3(\boldsymbol{r}-\boldsymbol{r'})\, d^3\boldsymbol{r'} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}$$

Where we have used the property of the delta function:

$$\int \delta(a-x)f(x)\, dx = f(a)$$

Thus we have found that the divergence of the electric field is equal to the charge density. This is one of Maxwell's Equations:

$$\boxed{\nabla \cdot \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}}$$

This gives us that positive charges are sources, and negative charges are sinks. When we don't have a charge there, the electric field just passes the point by, with no change.

We have seen what the divergence of a vector field can give us, but we also have the curl of a vector field:

$$\nabla \times \boldsymbol{E} = \nabla \times \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{r})\frac{\boldsymbol{R}}{R^2}\, d^3\boldsymbol{r}$$

We make the same assumption we made last time, that we can move the curl across the integral:

$$\frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{r})\frac{\nabla \times \boldsymbol{R}}{R^2}\, d^3\boldsymbol{r}$$

Recalling that $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$, we can set $\boldsymbol{r'} = \boldsymbol{0}$ for now, and then restore it later, since we know that the curl does not care about $\boldsymbol{r'}$. Now computing the curl:

$$\nabla \times \frac{\boldsymbol{r}}{r^3} = \boldsymbol{0}$$

This gets us that no matter what charge distribution we have:

$$\boxed{\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0}$$

1.2 Electric Potential

Let us now invoke Stokes' theorem. Suppose we have a vector field. If we want to pick a path between two points a and b, picked arbitrarily, we can define a path element $d\boldsymbol{l}$ along the path. If we compute the line integral

$$\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l}$$

If we then pick some other path back from b to a, not necessarily the same one, we have an integral over a loop:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = \oiint_{\text{surface}} \nabla \times \boldsymbol{E} \cdot d\boldsymbol{s}$$

This is Stokes' theorem. Since we know that the first integral is 0, we know that the integral given by Stoke's theorem is also 0. This means that we have path independence when picking curves through the electric field. The integral between points a and b have nothing to do with the way we move between them, but on the points a and b themselves. Thus the integral is dependent on some function at those two points:

$$\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = -[V(b) - V(a)]$$

This is known as the electric potential. Note that we only care about the difference in potential between two points, never the electric potential at a single point in space. Let us work out the potential for a particular case. Let us assume that we have a point charge with magnitude q. It produces an electric field given by

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q\hat{R}}{R^2}$$

We can compute the integral:

$$\int_a^b \frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2} \cdot d\boldsymbol{l}$$

Where $\boldsymbol{r'} = 0$, because we assume the charge is at the origin. If we use spherical coordinates, this integral is doable, and we find that we get

$$\int_a^b \frac{\hat{r}}{r^2} \cdot d\boldsymbol{l}$$

We note that $\hat{r} \cdot d\boldsymbol{l}$ is just dr, which gives us $-\frac{1}{r}\big|_a^b$, which gives us the $\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$ (where I have dropped some constants for now):

$$\int_a^b \frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2}\, d\boldsymbol{l} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

This tells us that

$$V(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r} + C$$

We also generally have the convention that potential is equal to 0 at infinity, $V(\infty) = 0$, which tells us that in this case $C = 0$, so

$$V(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}$$

Note that we can also go backwards, from the electric potential to the electric field:

$$\boldsymbol{E} = -\nabla V(\boldsymbol{r})$$

Thus, from the fact that $\nabla \times \boldsymbol{E} = 0$, we have found that $\boldsymbol{E} = -\nabla V$. We could also double check that $\nabla \times \nabla V = 0$. This is indeed true when we do the cross product.

We know that Gauss's law in differential form is

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

We also now know that $\nabla \times \boldsymbol{E} = 0$, and $\boldsymbol{E} = -\nabla V$. If we plug this second formula into the first one, we have that

$$\nabla \cdot \nabla V = -\frac{\rho}{\epsilon_0}$$

Now using the definition of ∇, we know that $\nabla \cdot \nabla = (\partial x)^2 + (\partial y)^2 + (\partial z)^2$. This is known as the Laplace operator, or the Laplacian, ∇^2. This tells us that:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

This is known as the Poisson Equation. This is sometimes difficult to solve, but one potential thing that we can do is care only about the locations where there is no charge, so the right side is 0, giving us the Laplace equation:

$$\nabla^2 V = 0$$

We can try to write down a formula for the electric potential for an arbitrary charge distribution ρ. For one charge, we had that

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

If we have a bunch of point charges, we just add the potential due to all of them:

$$V(\boldsymbol{r}) = \sum_i \frac{q_i}{4\pi\epsilon_0 R_i}$$

Where $\boldsymbol{R}_i = \boldsymbol{r} - \boldsymbol{r}_i$. Now how do we get this in terms of our charge density? We know that our charge density will be a sum of dirac deltas:

$$\rho(\boldsymbol{r}) = \sum_i q_i \delta^3(\boldsymbol{r} - \boldsymbol{r}_i)$$

We can then write out the potential:

$$V(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r'})}{4\pi\epsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|^2}\, d^3\boldsymbol{r'}$$

Let us talk a bit about the electric potentials. We have that $\boldsymbol{E} = -\nabla V$. We know that $\boldsymbol{F} = q\boldsymbol{E}$. This tells us that the force is $\boldsymbol{F} = -q\nabla V$. We know that work is given by $\boldsymbol{d} \cdot \boldsymbol{F} = -q\boldsymbol{d} \cdot \nabla V$. Let us think about units for a second. We know that work is measured in Joules. The right hand side has units of charge times units of the potential. From this, we find that the electric potential has units of J/C. This is called the Volt. Also note that the electric potential is different from the potential energy by units of charge. Take a proton for example. It has charge $e_p = -e_e$. This creates a potential,

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{e_p}{r}$$

Suppose we have an electron moving around the proton, and we want to find the potential energy of the electron, lets call this U. This is given by

$$U = e_e V = \frac{1}{4\pi\epsilon_0}\frac{e_e e_p}{r} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

Note that this is less than 0 (This is the binding energy of the Hydrogen atom).

Let us also look at boundary conditions. Let us consider a surface, with some charge density on the surface, σ. Given this boundary, what happens to the electric field and electric potential immediately above and below the boundary? In general, we'd say that this is too complicated. We note that if we zoom close enough to any surface, we can approximate it as a plane. We can then construct a Gaussian surface, and make a box that goes a bit above the charged surface and a bit below it. There are electric fields going out of the surface on both sides of the box. We can then integrate around the box:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{s}$$

We can ignore the sides of the box, since they are infinitely thin, so we have only the top and bottom. If the field above is called $\boldsymbol{E}_{\text{up}}$ and $\boldsymbol{E}_{\text{down}}$, and the box's surfaces have area A, we have that

$$\oint \boldsymbol{E} \cdot d\boldsymbol{s} = E_{\text{up}}A - E_{\text{down}}A = \frac{Q}{\epsilon_0} \to E_{\text{up}} - E_{\text{down}} = \frac{\sigma}{\epsilon_0}$$

Now recalling that $\boldsymbol{E} = -\nabla V$, we have that

$$-\left(\frac{\partial V}{\partial n}\right)_{\text{up}} + \left(\frac{\partial V}{\partial n}\right)_{\text{down}} = \frac{\sigma}{\epsilon_0}$$

This equation basically tells us that we have equipotential surfaces above and below the surface, and the difference between them is related to the surface charge density.

We have our electric field, and one equation tells us that

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

And we can look at the curl of the field:

$$\nabla \times \boldsymbol{E} = 0$$

These are the differential form of Maxwell's equations. We can find the integral form of the first equation:

$$\oiint \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{1}{\epsilon_0}Q_{enc} = \frac{1}{\epsilon_0}\int_S \rho \, dV$$

And the second of Maxwell's equations has integral form

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

When we look at boundary conditions, we often find that the integral formulation is easier to use. Suppose we have a surface, with some electric charge density σ. When we apply Gauss's Law in integral form, the important thing to do is to pick a Gaussian surface, a matchbox. This is a closed surface, with 6 sides, and thus we can apply the surface integral to it. We can make the area of the 4 sides very small, thus neglecting that section of the integral. We then only have to worry about the upper surface and the lower surface:

$$\oiint \boldsymbol{E} \cdot d\boldsymbol{s} = 0 + E_{up}^{\perp}A - E_{down}^{\perp}A = \frac{\sigma A}{\epsilon_0}$$

Where the 0 is the sides. This leaves us with

$$E_{up} - E_{below} = \frac{\sigma}{\epsilon_0}$$

We can now apply the second Maxwell equation. We use a line to represent the surface, and draw a rectangular closed path. Once again, we make this as thin as possible, making the sides of the rectangle have no action. Suppose the path has a width of l. The line integral will be

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = E^{||}_{up} \cdot l - E^{||}_{below} \cdot l = 0$$

This tells us that $E^{||}_{up} = E^{||}_{below}$. We have thus obtained two equations representing how the electric field changes in every direction as we pass through the boundary.

1.3 Electric Energy

If we have a bunch of charges, we generate an electric field $\boldsymbol{E}$. Suppose we have a charge Q, and we want to move it from point a to point b. In Newtonian mechanics, $W = \int_a^b \boldsymbol{F} \cdot d\,vecl$. The force necessary to move the charge through the electric field must be equal and opposite to the electric force:

$$\boldsymbol{F} = -\boldsymbol{F}_e$$

Thus we have that

$$W = -\int_a^b \boldsymbol{F}_e \cdot d\boldsymbol{l}$$

The electric force acting on the charge is the charge that we are moving times the electric field:

$$\boldsymbol{F}_e = Q\boldsymbol{F}_e$$

Thus we have that

$$W = -Q\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = -Q\left[-V(b) + V(a)\right] = Q\left[V(b) - V(a)\right]$$

Suppose we have a charge q_1, and we bring a charge q_2 from ∞ to some distance r_{12}. How much work do we do?

$$W = q_2\left[V(r_{12}) - V(\infty)\right] = q_2 V(r_{12}) = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$$

Note that if we look at the signs, we do match the fact that like charges repel and opposite charges attract.

Suppose we bring another charge q_3 in from ∞, to a position that is r_{13} away from q_1, and r_{23} away from q_2. We now have to add more terms:

$$W = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} + \frac{q_1 q_3}{4\pi\epsilon_0 r_{13}} + \frac{q_2 q_3}{4\pi\epsilon_0 r_{23}}$$

We see that with every charge that we bring in, we have to tack on a term for each previous charges' interaction with the new charge.

We now define the electric energy E:

$$\boxed{E = \frac{1}{4\pi\epsilon_0}\sum_{i<j}^{N} \frac{q_i q_j}{r_{ij}}}$$

This is the definition that uses the pairwise summation, but how do we do it with independent summations?

$$E = \frac{1}{4\pi\epsilon_0}\sum_{i,j=1,i\neq j}\frac{q_iq_j}{r_{ij}}$$

However, this is not correct, it double-counts, so we have to divide by 2:

$$E = \frac{1}{8\pi\epsilon_0}\sum_{i,j=1,i\neq j}\frac{q_iq_j}{r_{ij}}$$

We can mess with this a bit more:

$$E = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{N}q_i\left(\sum_{j=1,j\neq i}^{N}\frac{q_j}{r_{ij}}\right)$$

We note that the inner sum is the same as the electric potential generated by all the charges except for i, at the location of charge i:

$$E = \frac{1}{2}\sum_{i=1}^{N}q_iV_i$$

We can generalize this to a continuous charge distribution:

$$\boxed{E = \frac{1}{2}\int d^3\boldsymbol{r}\rho(\boldsymbol{r})V(\boldsymbol{r})}$$

We can apply this to a spherical charge distribution, with radius R, and uniform charge density ρ. The total charge is $Q = \int \rho\, dV$. We can ask how much work we have done to create this distribution of charges. We can use the formula, by first computing the potential generated by the spherical charge distribution:

$$V = -\int_{\infty}^{r}\boldsymbol{E}\cdot d\boldsymbol{l}$$

We can then insert this, and we would find that

$$E = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R}$$

We can ask how much work it would take to create an electron. Now we don't really know whether it is a uniform ball of charge, but let us model it as one. Let us also assume that the energy that we use is equal to the electron mass, $m_ec^2 = \frac{e^2}{4\pi\epsilon_0 R_e}$, where we have dropped the $\frac{3}{5}$, because we are not sure about the uniform charge distribution. We can then solve for the radius of the electron in our model:

$$R = \frac{e^2}{4\pi\epsilon_0 m_ec^2}$$

Now we note that $\frac{e^2}{4\pi\epsilon_0 c^2}$ is almost the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c^2} = \frac{1}{137}$$

Thus we have that $R \approx \frac{\alpha}{m_e}$, something known as the classical electron radius, and is about 2.7×10^{-15} meters, or 2.7 fm (fermi). Note that this model says that the electron is about the size of an atomic nucleus.

We can continue playing around with the energy expression. We know that

$$E = \frac{1}{2} \int \rho V(\boldsymbol{r})\, d^3\boldsymbol{r}$$

Now using Gauss's Law:

$$E = \frac{\epsilon_0}{2} \int \nabla \cdot \boldsymbol{E} V(\boldsymbol{r})\, d^3\boldsymbol{r}$$

Now we can use some nifty identities, and we use the chain rule:

$$(\nabla \cdot \boldsymbol{E})V(\boldsymbol{r}) = \nabla \cdot [\boldsymbol{E}V(\boldsymbol{r})] - \boldsymbol{E} \cdot \nabla V(\boldsymbol{r})$$

Now we note that $\nabla V = -\boldsymbol{E}$:

$$E = \frac{\epsilon_0}{2} \int \left(\nabla \cdot (\boldsymbol{E}V(\boldsymbol{r})) + \boldsymbol{E}^2\right)$$

Now we can use Gauss's Law to convert the first term to a surface integral:

$$\oiint EV\, d\boldsymbol{s}$$

Now since the surface is arbitrarily large, we just pick a surface at which the potential is 0, since we are so far away, so this entire term goes to 0. Thus we have that

$$\boxed{E = \frac{\epsilon_0}{2} \int \boldsymbol{E}^2\, d^3\boldsymbol{r}}$$

Suppose we have a proton and electron. Each generates an electric field, so the net field is given by

$$\boldsymbol{E} = \boldsymbol{E}_p + \boldsymbol{E}_e$$

We can insert this into the energy computation:

$$E = \frac{\epsilon_0}{2} \int (\boldsymbol{E}_p + \boldsymbol{E}_e)^2\, d^3\boldsymbol{r} = \frac{\epsilon_0}{2} \int \boldsymbol{E}_p^2 + \boldsymbol{E}_e^2 + 2\boldsymbol{E}_e \cdot \boldsymbol{E}_p\, d^3\boldsymbol{r}$$

We note that if we think of them as point particles, the first two terms go to ∞, so we pretend we don't see them and we look at the third. These terms are known as the self-energies of the electron and protons. The third term gets the attractive energy between the proton and the electron that we use when solving the Schrodinger equation. In Quantum Field Theory, we solve this issue using renormalization.

1.4 Conductors

A conductor is a piece of metal which contains free electrons. An electron being free means that in the presence of an electric field, it will move. We will have the simplest model of a conductor, which is essentially an electron gas. We will also not be considering time-varying conductors.

What are the interesting properties of a conductor in the electrostatic case. The first is that $\boldsymbol{E} = 0$ inside a conductor. If it were nonzero, the electrons would be moving.

Suppose we have a static electric field, and we bring a conductor in. The electrons will want to move, and it will drive positive charges to the side in the direction the field is pointing, and the negative charges will move against the field. The electrons cannot leave, so they will live on the surface. This creates an electric field inside the conductor, which exactly cancels the electric field outside. We essentially place charges on the surface, and maintain the fact that $\boldsymbol{E} = 0$ inside the conductor. According to Gauss's Law, $\rho = \epsilon_0 \nabla \cdot \boldsymbol{E} = 0$. Thus we have no charge in the middle of the conductor.

Suppose we take two points on the conductor, a and b, and compute the potential difference between them:

$$V(b) - V(a) = -\int_a^b \boldsymbol{E}\, d\boldsymbol{l} = 0$$

Thus everything in the conductor is at equal potential.

We also note that the electric field of the conductor is always normal to the surface of the conductor, there is no parallel component of the electric field. $\boldsymbol{E}_{||} = 0$, and $\boldsymbol{E}_{\perp} \neq 0$.

Let us now look at some examples. Suppose we have a conductor, and bring in a charge $+q$ some distance away from it. This will create an electric field, and will induce charges on the conductor. The negative charges will crowd to the side closest to the charge, and the positive charges will move to the opposite side. This will generate an attractive force between the charge and the conductor, since the negative charges are closer to the charge and thus will attract.

Let us look at a more interesting situation. Suppose we have some piece of conductor, with a hole in the center. If we insert a charge $+q$ in the middle of the hole, we can intuitively see that positive charges will go to the outside, and negative charges will go to the inside, closer to the charge. Note that we know the charge induced is $-q$, since we could create a Gaussian surface that includes just the inside edge, and we know that since $\boldsymbol{E} = 0$, $Q_{enc} = 0$, which can only be done if the induced charges are of magnitude $-q$. We also know that the total charge on the outside is $+q$, since the conductor has to be net neutral in terms of charge.

Suppose we have some oddly shaped conductor, with a cavity inside it. If we place some positive charges distributed on the outside of the conductor, we know that everything inside must have no electric field, including the cavity, no matter the outside conditions. This is a Faraday Cage.

Let us now look at what happens near the surface of the conductor. We have used Maxwell's equations to find boundary conditions, so let us apply these to metals.

We have an electric field outside the surface E_{up}, and the field below, E_{below}. Using the equation we derived, we have that

$$\boldsymbol{E}_{up} - \boldsymbol{E}_{below} = \hat{n}\frac{\sigma}{\epsilon_0}$$

In this case, we know that $\boldsymbol{E}_{below} = 0$, so we immediately find that

$$\boldsymbol{E}_{up} = \hat{n}\frac{\sigma}{\epsilon_0}$$

We also note that $\boldsymbol{E}_{up} = -\frac{\partial V}{\partial z}$, where we assume that the up direction is the $\hat{z}$ direction. We found this from the fact that $\hat{n} \cdot \boldsymbol{E} = -\nabla V$. Thus we have that

$$\frac{\partial V}{\partial z} = -\frac{\sigma}{\epsilon_0}$$

Let us now look at the force at the boundary. We have 3 fields, $\boldsymbol{E}_{up}$, $\boldsymbol{E}_{below}$, and $\boldsymbol{E}_{ext}$. We know that $\boldsymbol{E}_{up} = \boldsymbol{E}_{ext} + \boldsymbol{E}_{\sigma}$, where $\boldsymbol{E}_{\sigma}$ is the electric field generated by the surface charge. We also know that $\boldsymbol{E}_{below} = \boldsymbol{E}_{ext} - \boldsymbol{E}_{\sigma} = 0$. We also have the boundary condition, $\boldsymbol{E}_{up} - \boldsymbol{E}_{below} = 2\boldsymbol{E}_{\sigma} = \hat{n}\frac{\sigma}{\epsilon_0}$. From this, we find that

$$\boldsymbol{E}_{\sigma} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

Using the fact that $\boldsymbol{E}_{below} = 0$, we know that

$$\boldsymbol{E}_{ext} = \boldsymbol{E}_{\sigma} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

This relation gives how we determine what the charge density will be. The external field will have a force that acts on the surface charge. The force per unit area will be

$$\boldsymbol{F} = \sigma\boldsymbol{E}_{ext} = \frac{\sigma^2}{2\epsilon_0}\hat{n}$$

Note that this force will always try to rip charges away, no matter the sign of the charges, due to the σ^2. The electrons stay inside the conductor because of the work function of the metal that they are in, which keeps them bound to the surface. Note that this force has units of Newtons per unit area, which is defined as a Pascal in SI units. We can write the pressure in terms of the total electric field upwards:

$$\boldsymbol{F} = \frac{\epsilon_0}{2}\boldsymbol{E}_{up}^2$$

Capacitors

A basic capacitor is two pieces of metal, one side have a positive charge Q and the other having a negative charge $-Q$. We know that each piece of metal has a constant potential, V_1 and V_2 respectively. There is a potential difference between the two, $V = V_1 - V_2$. Suppose we increase the charge on both plates by a factor of two. What happens to the potential difference? We can look at Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

From this, we can see that if we double ρ, the potential will also double. We thus note that $\frac{Q}{V}$ has nothing to do with the electrostatics. We define this to be the capacitance C.

How much energy can a capacitor store? We can think of moving a bit of charge dq from one conductor to another, and the work that we do is equal to $sW = V\,dq$. We also know that $dW = \frac{Q}{C}\,dq$. We can then compute the work stored:

$$W = \int dW = \frac{1}{C}\int_0^Q q\,dq = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

We can think of this as the potential energy (as we see in RLC circuits).

1.5 Solving Laplace's Equation

Inside a conductor, we have that $\rho = 0$. From Maxwell's first equation, we know that $\nabla \cdot \boldsymbol{E} = 0$. We can then substitute the definition of the potential, $\boldsymbol{E} = -\nabla V$:

$$\nabla^2 V = 0$$

We will now look at how to solve this via separation of variables. Firs,t we'd like to note that if we know $\nabla^2 V = 0$ in volume Ω, and we know the boundary conditions on $\partial\Omega$, then the solution to Laplace's equation is unique. If we try a solution and it works and it obeys the boundary conditions, then it is the only solution to the equation.

Let us begin in Cartesian coordinates. Suppose we have an infinitely long box, and we are looking at one face of it. It has height a, and we know that the potential at the top and bottom rim is 0, and the potential on the side rims is given by some $V_0(y)$. We can write down the boundary conditions:

$$V(y=0) = V(y=a) = 0 \quad V(x=0,y) = v_0(y) \quad V \to 0 \text{ as } x \to \infty$$

Note that we can collapse ∇^2 down to the 2d case, where we ignore the z direction. Thus we have the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We will assume that the solution is of the form $V(x,y) = X(x)Y(y)$:

$$\frac{\partial^2}{\partial x^2}(X(x)Y(Y)) + \frac{\partial^2}{\partial y^2}(X(x)Y(Y)) = 0$$

We can then rewrite this:

$$\frac{d^2 X}{dx^2} Y + X \frac{d^2 Y}{dy^2} = 0$$

We can divide both sides by $X(x)Y(y)$:

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2}$$

Now this is where we note that X and Y are independent of each other, and thus the only way this can be satisfied is if both sides are constant. Thus we have reduce our partial diffeq of 2 variables into two ODEs. There are 3 cases for the constant λ, $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. Let us first consider the case where $\lambda < 0$. Let $\lambda = -\mu^2$:

$$\frac{d^2 Y}{dy^2} = \mu^2 Y \to Y = Ae^{\mu y} + Be^{-\mu y}$$

We can now evaluate boundary conditions. The first condition tells us that $Y(0) = 0$, which tells us that $A + B = 0$, or $A = -B$:

$$Y(y) = A(e^{\mu y} - e^{-\mu y})$$

The second condition tells us that $Y(y = a) = 0$, which we can insert and we are required to have that $e^{2\mu a} = 1$, or $2\mu a = 0$. This cannot be satisfied, since $\lambda = -\mu^2 \neq 0$. The only other way for this to be true is that $A = 0$, which cannot be true. Thus we see that the case where $\lambda < 0$ is not a valid solution.

Let us now try the case where $\lambda = 0$. This tells us that

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

This tells us that $Y(y) = Ay + B$. Inserting in boundary conditions, we know that $Y(0) = 0 \to B = 0$. This gets us that $Y(y) = Ay$, and the second condition leaves us with $Y(a) = 0 \to Aa = 0$, which once again does not work out, we cannot have $A = 0$.

All we are left with is the case where $\lambda > 0$:

$$\frac{d^2Y}{dy^2} = -\mu^2 Y$$

This gets us $Y(y) = A\sin(\mu y) + C\cos(\mu y)$. We can once again push through the boundary conditions. The first condition tells us that $Y(0) = 0$, which gives us that $B = 0$, so we have that $Y(y) = A\sin(\mu y)$. We can insert the second condition, $Y(a) = 0$, which gets us that $A\sin(\mu a) = 0$. This tells us that $\mu a = n\pi$ for $n = 1, 2, 3, \ldots$. This gets a family of solutions so far:

$$Y_n(y) = c_n \sin\left(\frac{n\pi y}{a}\right)$$

We note that any combination of these must be a solution, so we have that

$$Y(y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi y}{a}\right)$$

We can now consider the X solution:

$$\frac{1}{X}\frac{d^2X}{dx^2} = \left(\frac{n\pi}{a}\right)^2$$

This tells us that

$$X(x) = De^{fracn\pi xa} + Ee^{\frac{n\pi x}{a}}$$

We can use the fact that as $x \to \infty$, $V \to 0$, so this means that $E = 0$. Thus we have that

$$V_n(x,y) = c_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) \to V(x,y) = \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

Now we need to insert the third boundary condition, which states that $V(x = 0, y) = V_0(y)$. This tells us that

$$V_0(y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi y}{a}\right)$$

From here we use the identity that

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right)\, dy = \frac{a}{2}\delta_{nn'}$$

Where we are essentially leveraging the orthogonality of sine. Alternatively, this could be thought of as using the sine function as a unit vector, and computing an inner product.

Using this identity, we can multiply both sides by $\sin\left(\frac{n\pi y}{a}\right)$, and integrate:

$$\int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)\,dy = \sum_{n=1}^{\infty} c_n \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)\,dy$$

$$\int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)\,dy = c_{n'}\frac{a}{2}$$

We can then solve for the general coefficient:

$$c_n = \frac{2}{a}\int_0^a dyV_0(y)\int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)$$

Let us move on to doing this in spherical coordinates. The first step is to rewrite the Laplacian in spherical coordinates:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

We will restrict ourselves to cases with azimuthal symmetry, $\frac{\partial V}{\partial\phi} = 0$. Let us now assume our solution is of the form $V(r,\theta) = R(r)\Theta(\theta)$:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

Once again, we can use the same reasoning as previously, and set each separate equation equal to a constant, which we will call $l(l+1)$:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)$$

We can expand this out:

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - Rl(l+1) = 0$$

The general solution to this form of equation is a polynomial of order n, and in this case we have

$$R_l = Ar^l + \frac{B}{r^{l+1}}$$

Now considering the Θ equation:

$$\frac{d}{d\Theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\Theta$$

The general solution to equations of this form are Legendre polynomials, $P_l(\cos\theta)$, where $l = 0, 1, 2, \ldots$. We can then combine the two solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

This is the most general solution that we can have, with azimuthal symmetry.

Note that we have a similar orthogonality identity for the Legendre polynomials:

$$\int_{-1}^{1} dx\, P_l(x)P_{l'}(x) = \frac{2}{2l+1}\delta_{l\,l'}$$

In this instance, we have that $x = \cos\theta$:

$$\int_{-1}^{1} d(\cos\theta)P_l(\cos(\theta))P_{l'}(\cos\theta) = \int_0^{\pi} \sin\theta P_l(\cos\theta)P_{l'}(\cos\theta)\, d\theta$$

Let us do an example. Suppose we have a hollow sphere with surface charge $v_0(\theta)$, and radius R. This tells us that $V(R,\theta) = V_0(\theta)$. The question is to find the potential inside the sphere. Looking at the general solution, and noting that the origin is inside the sphere, we need the r^{l+1} terms to not diverge, so we know that $B_l = 0$. This leaves us with

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

We can insert the value $r = R$:

$$V_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

We can use the same trick as before, this time using the Legendre polynomial orthogonality:

$$\int_0^{\pi} d\theta \sin\theta V_0(\theta)P_{l'}(\cos\theta) = \sum_{l=0}^{\infty} A_l R^l \int_0^{\pi} d\theta \sin\theta P_l(\cos\theta)P_{l'}(\cos\theta)$$

$$A_{l'} = \frac{2l'+1}{2R^{l'}} \int_0^{\pi} \sin\theta V_0(\theta)P_l(\cos\theta)\, d\theta$$

Note that the solutions to Laplace's equation have a neat property that the value at a particular point is equal to the average of all points at a certain distance away from that point. The solutions to Laplace's equation are called harmonic functions. Another neat property is that there is no local minima or maxima, except at boundaries. This comes directly from the previous property.

Theorem 1.1 (Uniqueness Theorem 1)**.** *If we have a volume $\mathcal{V}$, with no enclosed charge, and some boundary $\mathcal{S}$, and the potential on the surface is given, then $V(\boldsymbol{r})$ is unique.*

Theorem 1.2 (Uniqueness Theorem 2)**.** *If we have some volume $\mathcal{V}$, with some enclosed charges, whose total charge is specified, the solution $V(\boldsymbol{r})$ is unique.*

1.6 Method of Images

Suppose we have some positive charge $+q$, with an infinitely large conducting plane below it, a distance d below to be specific. We want to calculate the electric field above the plane.

We begin by noting that the potential on the plane must be 0 everywhere. The trick for this problem is to imagine there is a negative charge $-q$ a distance d below the plane, and then imagine the plane didn't exist. We can now compute the potential at some arbitrary point (x, y, z), due to the two point charges:

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Now we note that (by inspection) the potential will always be 0 when $z = 0$, the same as if the plane was there. This therefore must be our solution, due to the uniqueness theorem. Since this satisfies the boundary condition, and it solves Laplace's equation in the region we care about, this must be the only solution. Note that from this, we can solve for the charge distribution on the metal, by using the fact that $\sigma = \epsilon_0 E_z$, and using the fact that $E_z = -\frac{\partial V}{\partial z}$. We can also solve for the total charge on the surface, via the newly acquired charge distribution. From that, we find that the induced total charge is $-q$, something that we could guess, from the fact that we expect all the electric field lines to terminate on the surface of the conductor.

Let us say we have 2 conductors, one placed at an angle θ with respect to the other, forming a wedge. We insert a charge somewhere in the wedge formed, and we want to find the locations of image charges to allow us to drop the conductors. This is a much trickier, and we can put down multiple image charges, and we can create images of images, and so on and so forth. From here, we have to tweak things to make it so that the series isn't infinite.

Let us graduate from conducting planes to conducting spheres. Suppose we have a metal sphere of radius R, with some charge q some distance a away from the center of the sphere. The metal sphere has potential $V = 0$ everywhere. We can actually solve this with a single image charge. We can put an image charge q' a distance b away from the center of the sphere, where $b < R$. Now we need to verify whether the boundary condition is met. Let us take a point $\boldsymbol{r} = (x, y, z)$. The potential generated at that point is given by the sum of the potentials of the two charges. Let $\boldsymbol{r}_1$ be the vector between the original charge and our point, and $\boldsymbol{r}_2$ be the vector between the image charge and our point:

$$\boldsymbol{r}_1 = (x, y, z-a) \quad \boldsymbol{r}_2 = (x, y, z-b)$$

From this, we can define our potential:

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_1} + \frac{q'}{r_2}\right]$$

We need this to satisfy the boundary conditions. First, we look at only 2 points on the sphere, $r = (0, 0, R)$ and $r = (0, 0, -R)$. We now want to determine the values of our charge q' and the distance b from the center we placed the image.

$$V(0,0,R) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{a-R} + \frac{q'}{R-b}\right] = 0$$

We can also look at the other point:

$$V(0,0,-R) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{a+R} + \frac{q'}{R+b}\right] = 0$$

We now want to solve these two equations. We can rewrite the first one as

$$(R-b)q = -q'(a-R)$$

From the second equation, we have that

$$(R+b)q = -q'(a+R)$$

We can then add the two equations:

$$Rq = -q'a \to q' = -\frac{R}{a}q$$

We can then solve for b, and we find that

$$b = \frac{R^2}{a}$$

We now want to see if this satisfies the potential condition for the entire sphere. We can write out the potential that we have

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-a)^3}} + \frac{-\frac{Rq}{a}}{\sqrt{x^2+y^2+\left(z-\frac{R^2}{a}\right)^2}}\right]$$

Now let us insert the fact that $x^2+y^2+z^2 = R^2$, and insert this into our potential:

$$V = \frac{1}{4\pi\epsilon}\left[\frac{q}{\sqrt{R^2+a^2-2az}} - \frac{\frac{Rq}{a}}{\sqrt{R^2-\frac{R^4}{a^2}-\frac{2zR^2}{a}}}\right]$$

Let us now move the $\frac{R}{a}$ in the numerator down into the denominator, which means that we divide by $\frac{R^2}{a^2}$, so we are left with the same denominator as the left term. Thus we have that

$$V_{\text{sphere}} = 0$$

Thus, our potential function is correct, and our image charge worked.

1.7 Multipole Expansion

This is an approximation method, similar to perturbation theory. This is used very frequently in theoretical physics. Suppose we have a lump of charge. We want to know what the potential is at some point $\boldsymbol{r}$ far away from our lump of charge. In general, we know that the answer is going to be

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r'})\,d\boldsymbol{r'}}{R}$$

Where $R = |\boldsymbol{r}-\boldsymbol{r'}|$. The presence of this R makes the integral much harder to calculate. If we think about a very far point, we can approximate R to discard the $\boldsymbol{r'}$, because we have that $\boldsymbol{r'}$ is going to be pretty close to the origin. Thus we say that $R \sim r$. Now we have that

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 r}\int \rho(\boldsymbol{r'})\,d\boldsymbol{r'}$$

This integral is just the total charge Q:

$$V(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0 r}$$

This is just the potential of a point-like charge. We expected this, since we are far away from the charge distribution. Think about looking at a star in the sky, which are these huge spheres, but we can approximate them as points. But what if the total charge is equal to 0? We know that the potential is not necessarily 0 far away, so we have to improve our approximation. We say that R is no longer approximately r, but we instead look at the definition:

$$R = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

We know that r is very big:

$$R = r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\theta}$$

Now we note that we have a function of the form $\sqrt{1+\epsilon}$, which has the Taylor expansion:

$$\sqrt{1+\epsilon} = 1 + \frac{1}{2}\epsilon + \mathcal{O}(\epsilon) + \dots$$

Using this, we have that $x = -2\frac{r'}{r}\cos\theta$

$$R \approx r\left(1 - \frac{r'}{r}\cos\theta\right) = r - r'\cos\theta$$

Thus we can rewrite our integral:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r'})\,d\boldsymbol{r'}}{r - \cos\theta r'}$$

This might seem hard, but we can again Taylor expand, using the fact that $\frac{1}{1+\epsilon} = 1 - \epsilon + \dots$:

$$= \frac{1}{4\pi\epsilon_0 r}\int \left(1 + \cos\theta\frac{r'}{r}\right)\rho(\boldsymbol{r'})\,d\boldsymbol{r'}$$

We can split this integral:

$$= \frac{1}{4\pi\epsilon_0 r}\left[Q + \frac{1}{r}\int r'\cos\theta\rho(\boldsymbol{r'})\,d\boldsymbol{r'}\right]$$

Now noting that we can write $\cos\theta = \hat{r}\cdot\hat{r}'$:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 r}\left[Q + \frac{1}{r}\int \boldsymbol{r'}\cdot\hat{r}\rho(\boldsymbol{r'})\,d\boldsymbol{r'}\right] = \frac{1}{4\pi\epsilon_0 r}\left[Q + \frac{\hat{r}}{r}\int \hat{r}\rho(\boldsymbol{r'})\,d\boldsymbol{r'}\right]$$

We define the dipole moment $\boldsymbol{p}$ to be the integral on the right:

$$\boldsymbol{p} = \int \boldsymbol{r}\rho(\boldsymbol{r'})\,d\boldsymbol{r'}$$

Now if we have the case where $Q = 0$:

$$V(\boldsymbol{r}) = \frac{\hat{r}\cdot\boldsymbol{p}}{4\pi\epsilon_0 r^2}$$

The dipole moment is the weighted position of the charge, rather than just the charge. The question is whether or not this depends on the coordinates we use. Suppose we have two different coordinate systems, $\boldsymbol{r}$ and $\boldsymbol{r'}$. The origins of the two coordinate systems differ by some vector $\boldsymbol{a}$. Thus,

$\boldsymbol{r'} = \boldsymbol{r} + \boldsymbol{a}$. We note that the charge densities must be the same, $\rho(\boldsymbol{r'}) = \rho(\boldsymbol{r})$. If we compute the dipole moment in both coordinate systems:

$$\boldsymbol{d} = \int \rho(\boldsymbol{r})\boldsymbol{r}\, d\boldsymbol{r}$$

$$\boldsymbol{d'} = \int \rho(\boldsymbol{r'})\boldsymbol{r'}\, d\boldsymbol{r'}$$

We want to find how these differ. It turns out that $\boldsymbol{d'} = \boldsymbol{a}Q + \boldsymbol{d}$. We can find this by taking the integral for $\boldsymbol{d'}$, and using the definition of $\boldsymbol{r'}$:

$$\boldsymbol{d'} = \int \rho(\boldsymbol{r})(\boldsymbol{r} + \boldsymbol{a})\, d\boldsymbol{r} = Q\boldsymbol{a} + \boldsymbol{d}$$

Where we have used that the charge density returns the same value. Note that we have a "sweet spot", which is when $Q = 0$. In this case, the dipole moment is the same from any coordinate system.

Let us do an example of computing the dipole moment of a system. Suppose we have two point charges ($+q$ at $\boldsymbol{r}_1$ and $-q$ at $\boldsymbol{r}_2$). We can write the charge distribution as

$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r} - \boldsymbol{r}_1) - q\delta(\boldsymbol{r} - \boldsymbol{r}_2)$$

We can write out the integral for the dipole moment, and we find that it is given by $q(\boldsymbol{r}_1 - \boldsymbol{r}_2)$.

What if both the total charge and the dipole moment are 0? Then we can expand to the quadropole moment. For example, suppose we have 4 charges, arranged at the corners of a square of length a, with alternating charges, $+q$ and $-q$. The net charge is 0, and the sum of the dipole moments is 0. (If we wanted to generalize this, we can create a cube, and we have the octopole). For the quadropole, we can look at the $\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|}$, which we can expand. Mathematicians have worked out a nice formula for this when $r \gg r'$:

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

We can insert this into the potential integral:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{r'})\, d\boldsymbol{r'}\, \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

We can now interchange the sum and the integral:

$$V(\boldsymbol{r}) = \sum_{n=0}^{\infty}\frac{1}{4\pi\epsilon_0 r^{n+1}}\int \rho(\boldsymbol{r'})\, d\boldsymbol{r'}\left[P_n(\cos\theta)r'^n\right]$$

Let us look at the quadropole, which is $n = 2$:

$$V_2(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 r^3}\int \rho(\boldsymbol{r'})P_2(\cos\theta)r'^2\, d\boldsymbol{r'}$$

1.8 Dielectrics

Let us begin with a neutral atom in an external field. This external field induces a dipole moment, $\boldsymbol{p} = \alpha \boldsymbol{E}$, where α is the polarizability, a scaling factor that is based on how "hard" it is to move around the charges. Different atoms have different polarizabilities, for example, Hydrogen has a scaled polarizability of $\frac{\alpha}{4\pi\epsilon_0} = 0.667 \times 10^{-30}\ m^3$. For Helium, we have $.205 \times 10^{-30}\ m^3$. Lithium has $24.3 \times 10^{-30}\ m^3$. Now our goal will be to make a model that can calculate the polarizability for an atom.

Suppose we model an atom as a positive charge q, surrounded by a uniformly charged sphere of electrons, of radius a. Let us put this system into an external electric field $\boldsymbol{E}$. The positive charge will shift away from the center, and let us say it moves a distance $\boldsymbol{d}$ away. The positive charge sees two forces, the force from the external field:

$$F_{ext} = qE$$

It will also feel the force due to the negative charges. We can use Gauss's Law to calculate the electric field a distance $\boldsymbol{d}$ away from the center of the sphere. We know that $\oint \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{Q}{\epsilon_0}$. In this case, the left side is

$$E_- \cdot 4\pi d^2 = \frac{1}{\epsilon_0}\frac{4}{3}\pi d^3 \frac{3}{4\pi a^3} q \to E_- = \frac{qd}{4\pi\epsilon_0 a^3}$$

We know that these two forces must balance out:

$$E_- = E$$

From this, we can calculate d:

$$d = \frac{4\pi\epsilon_0 a^3 E}{q}$$

From this, we can compute the dipole moment:

$$\boldsymbol{p} = \boldsymbol{d}q = 4\pi\epsilon_0 a^3 \boldsymbol{E}$$

Thus we see that the dipole moment is indeed $\alpha\boldsymbol{E}$, and we define $\alpha = 4\pi\epsilon_0 a^3$. Note that a is the radius of the atom, and if we plug in the Bohr radius of Hydrogen, we find that our polarizability is about $0.125 \times 10^{-30}\ m^3$, which is a pretty okay approximation.

Now suppose we have some arbitrary molecule, which can have any strange shape we want. We can pick a coordinate system, and split the electric field into 3 directions, and we can split our dipole moment into the same 3 directions. From this, we will have something of the form

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{zx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

This matrix is known as the polarizability tensor. This represent how polarizable the molecule is in different directions. There is a choice of coordinates such that the polarizability tensor is diagonalizable (since the matrix will be Hermitian):

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & 0 & 0 \\ 0 & \alpha_{yy} & 0 \\ 0 & 0 & \alpha_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Where here x, y, and z are not the same as they were in the previous expression. This is known as the intrinsic coordinate system.

We assumed that the system had no dipole moment before the external electric field, now let us assume that the molecule has a dipole moment to begin with. For example, let us look at water, which has a dipole moment. Suppose we apply an electric field in a direction that is not the same as the dipole moment. We can model the dipole as a positive charge q and a negative charge $-q$ separated by a distance d. If the electric field is uniform, we can see that there will be no net force, since the force on the negative charge is $-q\boldsymbol{E}$ and the force on the positive charge will be $q\boldsymbol{E}$, and these will cancel. However, there will be a torque, $\boldsymbol{N}$:

$$\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}$$

We can compute all of the torques and add them up:

$$\boldsymbol{N} = \boldsymbol{r}_+ \times \boldsymbol{F}_+ + \boldsymbol{r}_- \times \boldsymbol{F}_- = \boldsymbol{r}_+ \times q\boldsymbol{E} - \boldsymbol{r}_- \times q\boldsymbol{E} = q(\boldsymbol{r}_+ - \boldsymbol{r}_-) \times \boldsymbol{E} = q\boldsymbol{d} \times \boldsymbol{E}$$

Now we note that the left side is just the dipole moment:

$$\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E}$$

The torque will cause the dipole to align with the field. Intuitively, if we have a material and we place an external field onto it, we will have that the net dipole moment will begin to point in the direction of the field, based on the field strength:

$$\boldsymbol{P}_{tot} \sim \boldsymbol{E}$$

We can compute the energy of the dipole system:

$$U = qV(\boldsymbol{r}_+) - qV(\boldsymbol{r}_-) = q(V(\boldsymbol{r}_+) - V(\boldsymbol{r}_-)) = q\left[V(\boldsymbol{r}_+ + \boldsymbol{d}) - V(\boldsymbol{r}_-)\right]$$

We can approximate the inner portion as $\boldsymbol{d} \cdot \nabla V$:

$$U = q\boldsymbol{d} \cdot \nabla V = -q\boldsymbol{d} \cdot \boldsymbol{E} = -\boldsymbol{p} \cdot \boldsymbol{E}$$

Suppose we have a chunk of material with a dipole density $\boldsymbol{P}(\boldsymbol{r})$, a distribution of the dipoles of the individual molecules inside the material. Luckily, everything is linear so we can always add everything up. If we have one dipole, the potential it produces is given by

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p} \cdot \hat{r}}{r^2}$$

We can now sum over all dipoles:

$$V(\boldsymbol{r}) = \sum_i \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}_i \cdot \hat{r}_i}{r_i^2} = \frac{1}{4\pi\epsilon_0}\int d^3\boldsymbol{r'}\,\frac{\boldsymbol{P}(\boldsymbol{r'}) \cdot \hat{R}}{R^2}$$

Where we have as usual defined $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$.

We can compute $\nabla' \frac{1}{R} = \frac{\hat{R}}{R^2}$, and insert this:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\boldsymbol{r'}\left(\nabla'\frac{1}{R}\right) \cdot \boldsymbol{P}(\boldsymbol{r'})$$

We can now use integration by parts:

$$= \frac{1}{4\pi\epsilon_0} \int d^3\boldsymbol{r'} \left[\nabla' \left(\frac{\boldsymbol{P}(\boldsymbol{r})}{R} \right) - \frac{1}{R} \nabla' \cdot \boldsymbol{P}(\boldsymbol{r}) \right]$$

We can use the divergence theorem to convert the integral:

$$= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{S}} d\boldsymbol{S} \cdot \frac{\boldsymbol{P}(\boldsymbol{r'})}{R} - \frac{1}{4\pi\epsilon_0} \int d^3\boldsymbol{r'} \frac{-\nabla' \cdot \boldsymbol{P}(\boldsymbol{r'})}{R}$$

We can write these two terms after redefining $\sigma_b = \boldsymbol{n}_s \cdot \boldsymbol{P}(\boldsymbol{r})$ and $\rho_b = -\nabla \cdot \boldsymbol{P}(\boldsymbol{r'})$:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{\mathcal{S}} ds \cdot \frac{\sigma_b}{R} + \int_{\mathcal{V}} d^3\boldsymbol{r'} \frac{\rho_b(\boldsymbol{r'})}{R} \right]$$

We have converted a problem of dipole density into a problem about charge densities, which is confusing. Suppose we have a 1D chunk of material, with many dipoles. We can think of them laying head to tail, and the positive tails cancelling with the negative heads. The only ones that do not cancel are the ones at the far left and far right, which makes it seem like we have charges.

Now taking this to the volume case, we have the same argument, but instead we just generate a surface charge density. The reason we do this with dipoles instead of thinking of charges is that there are still dipoles there, they're the building block that we use to abstract it to charges. This explains the surface charge, but why is there a charge density? If the dipole density is uniform, we see that ρ_b vanishes. Only if the dipole density is nonuniform do we have an inner charge. This is intuitive, because the dipoles can no longer cancel exactly, and we have some excess charges.

We have covered the first case of dielectrics that we will look at, where $\boldsymbol{P}$ (the dipole density) is given. From this, we can calculate the electric potential produced by the dipole distribution:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\boldsymbol{P}(\boldsymbol{r'}) \cdot \hat{R}}{R^2} \, d^3\boldsymbol{r'}$$

Where as usual $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$. We have gone through and shown that this can be found via the surface charge $\sigma_b = \boldsymbol{n}_s \cdot \boldsymbol{P}(\boldsymbol{r})$ and the inner charge density $\rho_b = -\nabla \cdot \boldsymbol{E}$:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{\mathcal{S}} ds \cdot \frac{\sigma_b}{R} + \int_{\mathcal{V}} d^3\boldsymbol{r'} \frac{\rho_b(\boldsymbol{r'})}{R} \right]$$

From this, we can calculate the electric field generated, via the negative gradient of the potential. Note that the b subscript represents the fact that the charges are bound to the material.

We did an example where we took a dielectric sphere of radius R, and we can compute ρ_b and σ_b:

$$\rho_b = -\nabla \cdot \boldsymbol{P} = 0$$

$$\sigma_b = \hat{n} \cdot \boldsymbol{P} = \hat{r} \cdot \boldsymbol{P} = P \cos\theta$$

We see that there are no charges inside, and we have the top half of the sphere being positive and the bottom half being negative charges. We can then compute the electric field inside and outside:

$$\boldsymbol{E}_{in} = \frac{\boldsymbol{p}}{3\epsilon_0}$$

Where $\boldsymbol{p} = \frac{4}{3}\pi R^3 \boldsymbol{P}$. Outside, we expect the electric field to be the same as a dipole:

$$\boldsymbol{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\hat{r}}{r^3}$$

The second case is when we know all the free charges, which create an electric field that generates a dipole moment. Essentially, $\boldsymbol{P}$ is unknown. We know that we will have a bound charge density ρ_b, as well as a free charge density $\rho_f(\boldsymbol{r})$. By Maxwell's equations, we have that $\epsilon_0\nabla\cdot\boldsymbol{E} = \rho_b + \rho_f = -\nabla\cdot\boldsymbol{P} + \rho_f$. Moving unknowns to the left:

$$\nabla\cdot(\epsilon_0\boldsymbol{E} + \boldsymbol{P}) = \rho_f$$

This left unknown is called the electric displacement, $\boldsymbol{D}$:

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$$

From this, we can rewrite Maxwell's equation:

$$\nabla\cdot\boldsymbol{D} = \rho_f$$

Let us consider an example. We have a wire, with some free charges on it, given by line density λ_f. Suppose we then surround it with a cylinder of rubber. The electric field of the wire will generate some dipole density in the rubber, $\boldsymbol{P}$. We can get some information about this system via the electric displacement. We can write the differential form via the integral form:

$$\oiint \boldsymbol{D}\cdot d\boldsymbol{s} = \int \rho_f \, d^3\boldsymbol{r'}$$

We can make a Gaussian surface, a cylinder along the wire, of radius r and length L. We know that the $\boldsymbol{D}$ vector must be radially outward, so we have symmetry, and we can only care about the side, not the cylinder faces, because the normals are orthogonal to the $\boldsymbol{D}$ vectors. We can then write out the Maxwell equation:

$$\oiint \boldsymbol{D}\cdot d\boldsymbol{s} = (2\pi rL)D = \lambda L \rightarrow D = \frac{\lambda}{2\pi r}$$

This must be true inside and outside the rubber dielectric. We know that outside of the rubber, $\boldsymbol{D} = \epsilon_0\boldsymbol{E}$. From this, we have that the electric field outside of the rubber is given by

$$\boldsymbol{E}_{out} = \frac{\lambda}{2\pi\epsilon_0 r}$$

We note that the rubber made no difference, but this makes sense, because the rubber is net neutral, and thus the charge due to the wire is the only thing that matters. What about the inside? Inside, all we know is that

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E}_1 + \boldsymbol{P}_1 = \frac{\lambda}{2\pi r}$$

I denoted these with subscript 1s because this varies based on the properties/type of the rubber chosen.

Recall that when we had electric fields in free space, we had two conditions, $\nabla\cdot\boldsymbol{E} = \rho_f$ and $\nabla\times\boldsymbol{E} = 0$. However, in the case of $\boldsymbol{D}$, we only have the first condition. The second condition:

$$\nabla\times\boldsymbol{D} = \epsilon_0\nabla\times\boldsymbol{E} + \nabla\times\boldsymbol{P} = \nabla\times\boldsymbol{P}$$

Which is not necessarily 0. Thus we cannot ignore the presence of the dielectrics and act as if they do not affect $\boldsymbol{D}$.

Suppose we have a uniform dielectric sphere, with uniform $\boldsymbol{P}$. We have no free charges, so $\nabla \cdot \boldsymbol{D} = 0$. If we assume that $\nabla \times \boldsymbol{D} = 0$, we know that $\boldsymbol{D}$ must be either constant or 0, which cannot be true, because that would tell us that we have no electric field outside, which cannot be possible if we have a dipole density inside. Thus we have that $\nabla \times \boldsymbol{D} \neq 0$, which means that $\nabla \times \boldsymbol{P} \neq 0$. Intuitively, we know that this must be true, because the curl being 0 means that any path we make gives us an integral of 0. We can then cook up a path that cuts through the sphere. Outside, we expect it to be 0, but if we cut through the sphere, we know that it cannot be 0, because we have the uniform $\boldsymbol{P}$ inside the sphere.

The third case is when we don't know what the dipole is, but we can apply an electric field and measure the dipole moment generated (in materials known as linear dielectrics). Essentially, we supply the relationship between $\boldsymbol{E}$ and $\boldsymbol{P}$. In a dielectric, we have that

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}$$

This constant χ_e is known as the electric susceptibility, and it varies greatly based on the substance. Rewriting $\boldsymbol{D}$:

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon_0 (1 + \chi_e) \boldsymbol{E}$$

We define a new term, ϵ_r, known as the relative dielectric constant (vacuum is 1, air is 1.0003, water is 80.1, $KTaNbO_3$ has $34,000$):

$$\boldsymbol{D} = \epsilon_0 \epsilon_r \boldsymbol{E}$$

We once again rename constants, $\epsilon = \epsilon_0 \epsilon_r$, known as the permittivity of the substance:

$$\boldsymbol{D} = \epsilon \boldsymbol{E}$$

If we go back to the wire example, with rubber outside, we would need to know what the dielectric constant of the rubber would be, and then we can work out what $\boldsymbol{E}$ is from $\boldsymbol{D}$.

Suppose we have a conducting sphere of radius a, with total charge Q, surrounded by a dielectric shell, with dielectric constant ϵ and thickness b. We want to calculate the potential of the system. We need to solve for the electric field inside the shell, and outside of the shell.

We know that $\nabla \cdot \boldsymbol{D} = \rho_f$. We can also compute $\boldsymbol{D}$ outside of the dielectric:

$$\boldsymbol{D} = \frac{Q}{4\pi r^2} \hat{r}$$

Thus, outside of the dielectric ($r > b$), we know that

$$\boldsymbol{E} = \frac{Q}{4\pi \epsilon_0 r^2} \hat{r}$$

Now we can deal with the case where $a < r < b$. Here, we have that $\boldsymbol{E} = \frac{\boldsymbol{D}}{\epsilon}$, where the permittivity is different:

$$\boldsymbol{E} = \frac{Q}{4\pi \epsilon r^2} \hat{r}$$

We want to now get the electric field on the inside. We can compute the potential, which is the same anywhere on the conductor, we we have that

$$V_{metal} = -\int_{\infty}^{a} \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{\infty}^{b} \frac{Q}{4\pi\epsilon_0 r^2}\,dr - \int_{b}^{a} \frac{Q}{4\pi\epsilon r^2}\,dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)\right)$$

Which can be rewritten as

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{b} + \frac{1}{\epsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

We can now use the fact that $\boldsymbol{P} = \epsilon_0\chi_e\boldsymbol{E}$, and we have that

$$\rho_b = -\nabla \cdot \boldsymbol{P} = -\epsilon_0\chi_e\nabla \cdot \boldsymbol{E}$$

We can note by inspection that $\nabla \cdot \boldsymbol{E}$ must be 0, because our electric field is the field of a point charge, and anywhere other than the point charge we have a divergence of 0, because the charge density is 0 inside the dielectric. Thus we have that

$$\rho_b = 0$$

The dielectric has no volume charge density. We can then compute the surface charge density, $\sigma_b = \boldsymbol{P} \cdot \hat{n}$. For the outer surface, $\hat{n} = \hat{r}$:

$$\sigma_b^{outer} = \epsilon_0\chi_E\boldsymbol{E} \cdot \hat{r}\big|_{outer} = \epsilon_0\chi_e\frac{Q}{4\pi\epsilon b^2}$$

From this, the total charge induced on the outer surface is given by

$$\int \sigma_b\,dS = \frac{\epsilon_0\chi_e Q}{\epsilon_0}$$

From this, we know that the total induced charge on the inner surface will just be the opposite of that, since the dielectric is net neutral. We can find this explicitly:

$$\sigma_{inside} = \hat{n} \cdot \boldsymbol{P}$$

In this case, our choice of normal vector will be radially inwards, which gives us a negative sign:

$$\sigma_{inside} = -\epsilon_0\chi_e\frac{Q}{4\pi\epsilon a^2}$$

We can then calculate the total:

$$Q_{total}^{inner} = \int \sigma_{inner}\,ds = -\frac{\epsilon_0\chi_e Q}{\epsilon}$$

We see that indeed we have the exact opposite induced charge. Note that if we get rid of the dielectric, χ_e becomes 0, and thus we have no induced charges, as we expect. Also note that if we let $\chi_e \to \infty$, this allows the dielectric to approximate a conductor (recall that $\epsilon = \epsilon_0(1 + \chi_e)$). We can check this, by seeing if the electric field inside goes to 0. We have that

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

And we see that as $\chi_e \to \infty$, we have that $\boldsymbol{E} = 0$.

Let us do a slightly different problem. Suppose we have a parallel plate capacitor, with Q on one plate and $-Q$ on the other. We know that $Q = CV$, and $C = \frac{A\epsilon_0}{d}$. In real world applications, we fill the space in between the two plates with a dielectric. If we fill it with a material with dielectric constant ϵ, we want to find the new capacitance, which is where we replace ϵ_0 with ϵ, which increases the capacitance:

$$C = \frac{A\epsilon}{d}$$

We can derive this by either starting with the potential and calculating the charge, or giving the charge and calculating the potential. We can do the latter, where we know that

$$V = -\int \boldsymbol{E} \cdot d\boldsymbol{l}$$

The electric field is being passed through the dielectric, so the electric field must get smaller (recall that the denominator of the shell electric field had ϵ instead of ϵ_0, which leads to a smaller field magnitude). The dielectric opposes the electric field, because the surface charges of the dielectric oppose the electric field:

$$V_{di} = -\int \boldsymbol{E} \cdot d\boldsymbol{l} = V_0 \frac{\epsilon_0}{\epsilon}$$

We know that $C = \frac{Q}{V}$, so we have an increased capacitance, since the potential has gone down.

Recall that we can compute the energy density via

$$W = \frac{\epsilon_0}{2} \int \boldsymbol{E}^2 \, d^3\boldsymbol{r'}$$

How can we modify this to deal with dielectric materials? We can change the ϵ_0 to an ϵ:

$$W = \frac{\epsilon}{2} \int \boldsymbol{E}^2 \, d^3\boldsymbol{r'}$$

We can also do this with $\boldsymbol{D}$:

$$W = \frac{1}{2} \int \boldsymbol{D} \cdot \boldsymbol{E} \, d^3\boldsymbol{r'}$$

Suppose we have a capacitor, with some distance d between the two rectangular plates, and side lengths a and l, one plate with Q and the other with $-Q$. Suppose we now insert a dielectric material partway through the gap, but not fully into. It has side length a, and we insert it a distance $l - x$ into the capacitor. It turns out that there will be a force either pulling in the dielectric or repelling it. We know that the energy in the capacitor is given by

$$W = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C}$$

And we know that the capacitance will just be given by

$$C = C_{di} + C_{air} = \frac{ax\epsilon_0}{d} + \frac{a(l-x)\epsilon}{d}$$

Thus we have that the energy is a function of x. We can calculate the force from the energy via a derivative:

$$\boldsymbol{F} = -\nabla W$$

We can compute the force in the x direction via

$$F_x = -\frac{\partial W}{\partial x}$$

We have to be careful about what we consider fixed, we want to take the derivative with Q fixed, because fixing V would require something like a battery to move charges.

Thus we want

$$F_x = -\frac{d}{dx}\left[\frac{Q^2}{2}\frac{d}{ax\epsilon_0 + a(l-x)\epsilon}\right] = -\frac{\epsilon_0 \chi_e a}{2d}V^2$$

We see that this is indeed a pulling force, which makes sense, because when the dielectric goes in, it cancels the nearby electric field, and leads to a surface charge on the dielectric, which is then attracted to the charges to the left on the capacitor. This makes the dielectric move inwards.

Consider the example of a dielectric ball with constant ϵ, placed in an external electric field $\boldsymbol{E}_0$. We want to calculate everything that we can calculate. We can do this by solving Laplace's equation $\nabla^2 V = 0$, to get V_{inside} and $V_{outside}$, discarding higher order Legendre polynomials:

$$V_{inside} = A_1 r P_1(\cos\theta)$$

$$V_{outside} = -E_0 r\cos\theta + \frac{B_1}{r^2}P_1(\cos\theta)$$

We need 2 boundary conditions to determine the constants. One is that the potential is continuous, so we know that $V_{in}(r=R) = V_{out}(r=R)$. The second condition is that $\boldsymbol{D}^\perp$ is continuous:

$$\epsilon \boldsymbol{E}^\perp_{inside} = \epsilon_0 \boldsymbol{E}^\perp_{outside}$$

We know this because we have no free charges anywhere, so $\boldsymbol{D}^\perp_{in} = \boldsymbol{D}^\perp_{out}$. Now using the relationship between $\boldsymbol{E}$ and V, we know that

$$\epsilon\frac{\partial V_{in}}{\partial r} = \epsilon_0\frac{\partial V_{out}}{\partial r}$$

These two conditions get the fact that

$$\boldsymbol{E}_{inside} = \frac{3}{\epsilon_r + 2}\boldsymbol{E}_0$$

And

$$\boldsymbol{P} = \frac{\epsilon_0(3\chi_e)}{3+\chi_e}\boldsymbol{E}_0$$

2 Magnetostatics

In the early 1900s, we began to understand the basics of magnetism, and we began quantitative studies of the forces generates by moving charges. Let us begin with the concept of current. We know that if we have a charge q, moving with velocity $\boldsymbol{v}$, we produce a current $\boldsymbol{I}$, given by $\boldsymbol{I} = q\boldsymbol{v}$. We can think about the units of the current, we know that q is in terms of Coulombs, and $\boldsymbol{v}$ is in terms of meters per second. The definition of current is the amount of electric charge that moves through a point per unit time. Thus the proper unit should be Coulombs per second, known as the Ampere. $1A = 1\frac{C}{s}$.

Let's say we have a line charge, with line charge density λ, which has units of C/m. If this line is moving with a certain velocity $\boldsymbol{v}$, then the current will be $\boldsymbol{I} = \lambda\boldsymbol{v}$. We can also talk about a moving surface charge density σ, which generates a surface current, $\boldsymbol{K} = \frac{\boldsymbol{I}}{l}$. Finally, we have the "body" current density, generally given by $\boldsymbol{J}$, which is something like taking a small tube of area ds and measuring the amount of current flowing through the tube dI:

$$\boldsymbol{J} = \frac{d\boldsymbol{I}}{ds}$$

Let us say we have some flow of charges, and we have a volume, and we watch how the charges move around the volume, and how much charge flows out of the surface per unit time. We study this by taking a particular surface element $d\boldsymbol{s}$. The total charge inside the volume is some Q, and we want to know how that changes, $\frac{dQ}{dt}$. This change must come from the flow of the current through the surface. Per unit area, the change in charge is given by the current density dotted with the direction:

$$\frac{dQ}{dt} = -\oint\!\!\!\oint_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{s}$$

We know that total charge Q is given by the charge density integrated over the volume:

$$Q = \int_{\mathcal{V}} \rho \, d^3\boldsymbol{r}$$

From this, we have that

$$\frac{dQ}{dt} = \int_{\mathcal{V}} \frac{d\rho}{dt} \, d^3\boldsymbol{r}$$

We can now use Gauss's Law to do the current density integral:

$$\oint\!\!\!\oint_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{s} = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{J} \, d^3\boldsymbol{r}$$

Thus we have that

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} \, d^3\boldsymbol{r} = 0$$

This means that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

This is known as the continuity equation.

Suppose we have two wires, with current flowing through them in the same direction, it turns out that they attract each other, and when the flows are opposite directions, they repel each other. Thus we have a force due to the current flow. How do we know that the repelling and attracting forces are not due to electric effects? The reason is that we know the charge density is 0 in both wires. This is because the electron charges are exactly cancelled by the atomic nuclei charges:

$$\rho = \rho_{\text{electron}} + \rho_{\text{ion}} = 0$$

We have that ρ_{electron} is moving, which the ion density is not moving. Since both wires are electrically neutral, the current flow must be generating some other force, that isn't electric in nature. We can study the forces between the two currents, and we see that the force will be proportional to $\frac{I_1 I_2}{r^2}$, and we will assume that the forces are generated by magnetic fields:

$$\boldsymbol{F} = \boldsymbol{I}_1 \times \boldsymbol{B}_2$$

2.1 Biot-Savart Law

The Biot-Savart Law tells us the relationship between the magnetic field and the flowing current:

$$\boxed{\boldsymbol{B} = \frac{\mu_0}{4\pi}\int \frac{d\boldsymbol{I}\times\hat{R}}{R^2}}$$

Where μ_0 is the permeability of free space, and has the value of $4\pi\times 10^{-7}$ Newtons per square Ampere. From this, we have that the magnetic field has units of Teslas, which are Newtons per Ampere meter. This law is analogous to Coulomb's law in electrostatics.

Suppose we have a line of current I, and we want to find the magnetic field a point z away from the line. We can use the right hand rule, and we see that the magnetic field is constant on circles around the wire. We can calculate the magnetic field. We have some current element dI, that is l away from the perpendicular point. We want the cross product between $d\boldsymbol{I}$ and $\boldsymbol{R}$, which is from the current element to the point. We can use the definition of the cross product, $\boldsymbol{A}\times\boldsymbol{B} = AB\sin\theta$. In this case, θ is the angle between the perpendicular and the current element. We can then convert the cross product line integral to a scalar integral in terms of θ. We use the Pythagorean theorem to get $\boldsymbol{R}^2$:

$$B = \frac{\mu_0}{4\pi}\int \frac{I\,dl\sin\theta}{l^2+z^2}$$

We can redefine $\sin\theta$ to use the complementary angle, let's call it α, $\sin\theta = \sin(\pi-\alpha) = \sin\alpha$ (can find this via trig identities). We then note that $\sin\alpha = \frac{z}{\sqrt{l^2+z^2}}$:

$$\boldsymbol{B} = \frac{I\mu_0}{4\pi}\int_{-\infty}^{\infty}\frac{z\,dl}{(l^2+z^2)^{3/2}}$$

It turns out that this is equal to

$$B = \frac{I\mu_0}{2\pi r}$$

Now let us go back to having two currents. If we look at the force due to the first current on the second current, we expect it to go into the page. We observe that the force is to the left, and thus the force must be given by

$$\boldsymbol{F} = \boldsymbol{I}_2\times\boldsymbol{B}_1$$

We have to integrate over all distance, so that will be infinite, since the currents go infinitely. Instead, we assume that the wires are very long, of length L, and thus the force is given by

$$\boldsymbol{F} = LI_2\frac{I_1\mu_0}{2\pi} = L\frac{I_1I_2\mu_0}{2\pi}$$

We generally look at the force per unit length:

$$\frac{F}{L} = \frac{I_1I_2\mu_0}{2\pi}$$

Suppose we have $\boldsymbol{B}$ in space, and we put a charge in there, the charge will experience some current, $q\boldsymbol{v}$, and the current will feel a force, equal to $\boldsymbol{F} = \boldsymbol{I}\times\boldsymbol{B} = q\boldsymbol{v}\times\boldsymbol{B}$. This is known as the Lorentz force. We note that if the charge is not moving, we have no force. It turns out that this force will not do any work to the charge:

$$dW = d\boldsymbol{r} \cdot \boldsymbol{F} = d\boldsymbol{r} \cdot \left(q\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{B}\right) = d\boldsymbol{r} \times (q\frac{d\boldsymbol{r}}{dt} \cdot \boldsymbol{B}) = 0$$

Where the cross product between $d\boldsymbol{r}$ and $\frac{d\boldsymbol{r}}{dt}$ is 0.

Just to recap the Biot-Savart Law, we have 3 types of possible currents. We have body current $\boldsymbol{J}$, surface current $\boldsymbol{K}$, and the line current $\boldsymbol{I}$. We generally have a body current. We can then compute the magnetic field at some point $\boldsymbol{r}$:

$$B(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r'}) \times \hat{R}}{R^2}\, d^3\boldsymbol{r'}$$

And we have analogous integrals for the different types of current. We have that

$$d\boldsymbol{I} = \boldsymbol{I}(\boldsymbol{r})\, dl$$

We also did an example and showed that for a current carrying-wire, the magnetic field is aligned in concentric rings around the wire, and is of the form

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi r}$$

Now suppose we have a loop of current-carrying wire, of radius r, and we want to find the magnetic field above the center of the loop, a height z above. We can look at a particular current element on the loop, and it generates a $\boldsymbol{B}$ field, and we note that the opposite current element cancels the horizontal component of the magnetic field. Thus we have that the final $\boldsymbol{B}$ field will be strictly in the $\hat{z}$ direction. We can look at the Biot-Savart Law, and we need to compute the integral

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{I} \times \boldsymbol{R}}{r^3}\, dl$$

Where we only care about the z direction. We know that $\boldsymbol{I}$ has a component in the x direction, and since the orientation is such that the current is flowing into the page on the right, we have that $\boldsymbol{I} = (I, 0, 0)$, and we have that $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'} = (0, 0, z) - (0, a, 0) = (0, -a, z)$. Doing the cross product and looking at the z component, we have Ia. We also note that $R = \sqrt{z^2 + a^2}$, via the Pythagorean theorem. Thus we can write the integral as

$$\boldsymbol{B} = \frac{\mu_0 Ia}{4\pi(z^2 + a^2)^{3/2}}\int dl = \frac{\mu_0 Ia}{4\pi(z^2 + a^2)^{3/2}} 2\pi a = \frac{\mu_0 Ia^2}{2(z^2 + a^2)^{3/2}}$$

We can pull out the Ia because the field is symmetric, any current element will produce a field of strength Ia in the z direction.

Let us do an example of finding the magnetic field using the Biot-Savart Law. Suppose we have a circular loop of radius R, with current $I\hat{\phi}$, and we want to find the magnetic field at point $\boldsymbol{r} = z\hat{z}$. We can write out the current coordinates, $\boldsymbol{r'} = R\hat{r}' = R\cos\phi' + R\sin\phi'$. The Biot-Savart Law states that

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int dl' \frac{\boldsymbol{I} \times (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3}$$

We know that $\boldsymbol{r} - \boldsymbol{r'} = z\hat{z} - R\boldsymbol{r'}$. From this, we have that $|\boldsymbol{r} - \boldsymbol{r'}|^3 = (z^2 + R^2)^{3/2}$. Computing the cross product in the numerator:

$$\boldsymbol{I} \times (\boldsymbol{r} - \boldsymbol{r'}) = I\hat{\phi}' \times (z\hat{z} - R\hat{r}') = Iz(\hat{\phi}' \times \hat{z}) - IR(\hat{\phi}' \times \hat{r}') = Iz\hat{r'} + IR\hat{z'} = Iz(\cos\phi'\hat{x} + \sin\phi'\hat{y}) + IR\hat{z}$$

Finally, we have our line element, $dl' = Rd\phi'$. Thus we can write out our integral:

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int_0^{2\pi} d\phi' R \frac{Iz(\cos\phi'\hat{x} + \sin\phi'\hat{y}) + IR\hat{z}}{(R^2 + z^2)^{3/2}}$$

Now noting that when we integrate the sinusoids from 0 to 2π, they both disappear, we are left with

$$\boldsymbol{B} = \frac{\mu_0 R}{4\pi}\int_0^{2\pi} d\phi' \frac{IR\hat{z}}{2(R^2 + z^2)^{3/2}} = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}\hat{z}$$

2.2 Maxwell's Third Equation

Let us now look at the divergence of the $\boldsymbol{B}$ field. The divergence of an electric field was very important, it was related to the charge density. For $\boldsymbol{B}$:

$$\nabla \cdot \boldsymbol{B} = \frac{\mu_0}{4\pi}\int \nabla_{\boldsymbol{r}} \cdot \left(\frac{\boldsymbol{J}(\boldsymbol{r}) \times \boldsymbol{R}}{R^3}\right) d^3\boldsymbol{r'}$$

Where the subscript on the ∇ defines the vector that it is acting on. We can now switch the cross and the dot:

$$= \frac{\mu_0}{4\pi}\int (\nabla_{\boldsymbol{r}} \times \boldsymbol{J}) \cdot \frac{\boldsymbol{R}}{R^3}\, d^3\boldsymbol{r'}$$

We can now switch the order of the cross prduct terms, in exchange for a negative sign:

$$= \frac{\mu_0}{4\pi}\int -(\boldsymbol{J} \times \nabla_{\boldsymbol{r}}) \cdot \frac{\boldsymbol{R}}{R^3}\, d^3\boldsymbol{r'}$$

We can now swap the cross and dot back:

$$= \frac{\mu_0}{4\pi}\int -\boldsymbol{J} \cdot \left(\nabla_{\boldsymbol{r}} \times \frac{\boldsymbol{R}}{R^3}\right) d^3\boldsymbol{r'}$$

This internal cross product is 0, and thus the divergence of $\boldsymbol{B}$ is 0:

$$\boxed{\nabla \cdot \boldsymbol{B} = 0}$$

Intuitively, this is because the magnetic field always closes on itself, unlike the electric field. This is the third Maxwell Equation. Writing it in integral form:

$$\oiint_{\mathcal{S}} \boldsymbol{B} \cdot d\boldsymbol{s} = 0$$

2.3 Maxwell's Fourth Equation

Up next would be the curl of $\boldsymbol{B}$:

$$\nabla \times \boldsymbol{B} = \frac{\mu_0}{4\pi} \int \left(\nabla_{\boldsymbol{r}} \times \left(\boldsymbol{J}(\boldsymbol{r'}) \times \frac{\boldsymbol{R}}{R^3} \right) \right) d^3\boldsymbol{r'}$$

We can use vector calculus identities, and we can expand this out:

$$= \frac{\mu_0}{4\pi} \int \boldsymbol{J} \left(\nabla \cdot \frac{\boldsymbol{R}}{R^3} \right) - \left(\boldsymbol{J} \cdot \nabla \frac{\boldsymbol{R}}{R^3} \right) d^3\boldsymbol{r'}$$

Now we use the fact that $\nabla \cdot \frac{\boldsymbol{R}}{R^3} = 4\pi\delta^3(\boldsymbol{r} - \boldsymbol{r'})$, and the right term has a complicated explanation, but if you do an integration by parts, and we will take use of the fact that $\nabla \cdot \boldsymbol{J} = 0$ for a static system, and we are left with 0. Thus we have that

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}(\boldsymbol{r})$$

This is the fourth Maxwell Equation:

$$\nabla \cdot \boldsymbol{E} = \rho$$
$$\nabla \times \boldsymbol{E} = 0$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}(\boldsymbol{r})$$

Note that Maxwell's contribution to the fourth equation was to add another term to it, which allowed for EM waves, which we will see later.

We can switch from the differential form to the integral form:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}(\boldsymbol{r}) \to \oiint_{\mathcal{C}} \nabla \times \boldsymbol{B} = \mu_0 \oiint_{\mathcal{S}} \boldsymbol{J}(\boldsymbol{r})\, d\boldsymbol{s} \to \oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I$$

We generate a surface above a closed loop of current, and the magnetic field integrated along the rim of the hat is equal to $\mu_0 I$. This is Ampere's Law:

$$\boxed{\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I}$$

This is a magnetic version of Gauss's Law.

2.4 Using Ampere's Law

Suppose we have an infinitely long line, and we want to find the magnetic field at some point away from the line. We begin by realizing that the $\boldsymbol{B}$ field along a concentric circle around the wire will be constant, and we will integrate the $\boldsymbol{B}$ field along that line:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = B 2\pi r$$

And we know from Ampere's Law that this will be equal to $\mu_0 I$:

$$2\pi r B = \mu_0 I \to B = \frac{\mu_0 I}{2\pi r}$$

This is a much nicer way of doing that problem that we already did.

Let us do another example of using Ampere's Law. Suppose we have an infinitely long solenoid, a coiled wire, with some current $\boldsymbol{I}$ passing through it. The solenoid has a parameter that defines the number of coils per unit length. We want to find the magnetic field in the solenoid. Without proof, we will state that the field inside the loop will be constant, pointing upwards through the solenoid. It turns out that the magnetic field outside the solenoid is actually 0 everywhere. The magnetic field is entirely confined inside a solenoid. Let us now calculate the B field inside the solenoid. We make a rectangular integration path that has one edge inside and another edge outside the solenoid. Suppose that the path length parallel to the field is l:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = Bl + 0 + 0 + 0 = Bl$$

On the right hand side of Ampere's Law, we have $\mu_0 nlI$, where n is the number of turns per unit length. Thus we have that

$$B = \mu_0 nI$$

We can also make a solenoid torus, in which case the magnetic fields will be circular, inside the solenoid. This bypasses the effects of the ends of a straight solenoid. This can be used for applications like fusion, in Tokamak reactors.

Let us do another Ampere's Law problem. Suppose we have a block, from $z = -a$ to $z = a$, with current $\boldsymbol{J}$, coming out of the side, in the $+x$ direction. We want to find the magnetic field above, below, and inside the block. Inside the block, we can make an Ampere loop, of width l and height z. By Ampere's Law:

$$\oint d\boldsymbol{r} \cdot \boldsymbol{B} = \mu_0 \int d\boldsymbol{s} \cdot \boldsymbol{J}$$

We can break the loop into 4 chunks, and for two of those, the vertical ones, the field is 0. Since by the right hand rule, the magnetic field on the top surface goes to the left, and on the bottom surface it goes to 0, we can set up Ampere's Law as

$$2Bl = \mu_0 Jl(2z) \to B = \mu_0 Jz$$

From this, inside the block, we have that $\boldsymbol{B} = -\mu_0 Jz\hat{y}$.

On the top side of the block, we make an Ampere loop that encompasses the entire block, of height $2a$, and by Ampere's Law:

$$2Bl = \mu_0 Jl(2a)$$

And from this, we have that on the top side of the block:

$$\boldsymbol{B} = -\mu_0 Ja\hat{y}$$

and on the bottom side of the block

$$\boldsymbol{B} = \mu_0 Ja\hat{y}$$

2.5 Boundary Conditions

Suppose we have some surface, and we want to look at the boundary condition for the magnetic field across the boundary. Using Maxwell's third equation, $\nabla \cdot \boldsymbol{B}$, we can convert to integral form. We can make a Gaussian surface, with negligible side lengths, and we have that $B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$, because the net flux must be 0, by Maxwell's equation. We can use Maxwell's fourth law, by making a rectangular path with top and bottom length l, that contains the boundary. We have that $l(V^{||}_{\text{up}} - B^{||}_{\text{below}}) = \mu_0 I_{\text{total}}$, where the subtraction is due to the orientation of the path segments. The total current is given by the length of the path times the density, $I_{\text{total}} = lk$. Thus we have the two continuity equations:

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \qquad V^{||}_{\text{up}} - B^{||}_{\text{below}} = \mu_0 k$$

2.6 Magnetic Vector Potential

Since we often don't work directly with fields, we work with potentials, so we want to introduce some magnetic potential. We begin by looking at Maxwell's 3rd equation:

$$\nabla \cdot \boldsymbol{B} = 0$$

From this equation, we know that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, because $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$. (We can swap the cross and dot, and the cross of ∇ with ∇ is 0). We now insert this into Ampere's Law:

$$\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0 \boldsymbol{J}$$

This can be rewritten as

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

We note that there are many possible $\boldsymbol{A}$s that can fit the requirement for $\boldsymbol{B}$. Suppose we have $\boldsymbol{A}_1 \to \boldsymbol{B} = \nabla \times \boldsymbol{A}_1$. We can generate some $\boldsymbol{A}_2 = \boldsymbol{A}_1 + \nabla\phi$, where ϕ is any scalar field. We can then see that $\boldsymbol{B} = \nabla \times (\boldsymbol{A}_1 + \nabla\phi) = \nabla \times \boldsymbol{A}_2$. There are an infinite number of $\boldsymbol{A}$s that will give us $\boldsymbol{B}$. This is called the gauge degrees of freedom, or gauge symmetry.

Because of this property, we can impose conditions on $\boldsymbol{A}$, and find some $\boldsymbol{A}$ that will give us $\boldsymbol{B}$. For example, we can impose the condition that $\nabla \cdot \boldsymbol{A} = 0$ or that the third copmonent of $\boldsymbol{A}$ is 0, $\boldsymbol{A}^3 = 0$. These are called gauge conditions. Let us impose the condition that $\nabla \cdot \boldsymbol{A} = 0$. This is known as the Coulomb gauge (Coulomb actually didn't do this). In this case, the first term of the expression $\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$ goes away, and we have that

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

We can then solve for $\boldsymbol{A}$, which is known as the vector potential. This is very similar to Poisson's equation, except we have 3 equations, instead of 1. When we solve this, we will have that

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r'})\, d^3\boldsymbol{r'}}{R}$$

Once we have this, we can compute the curl of $\boldsymbol{A}$ to get $\boldsymbol{B}$. This is useful for complicated cases, where solving for $\boldsymbol{B}$ directly is difficult. One nontrivial example of this is to take a sphere of radius R, with uniform surface charge density σ. We then let it rotate with some angular velocity ω. This generates a current, and we want to compute the $\boldsymbol{A}$ inside and outside of that sphere. Using the solution for $\boldsymbol{A}$, we find that

$$\boldsymbol{A}_{\text{inside}} = \frac{\mu_0 R\sigma}{3}(\boldsymbol{\omega} \times \boldsymbol{r})$$

$$\boldsymbol{A}_{\text{outside}} = \frac{\mu_0 R^4\sigma}{3r^3}(\boldsymbol{\omega} \times \boldsymbol{r})$$

The derivation of this result is in the textbook. From these, we can compute the magnetic field:

$$\boldsymbol{B}_{\text{inside}} = \nabla \times \boldsymbol{A}_{\text{inside}} = \frac{2}{3}\mu_0 \sigma R \boldsymbol{\omega}$$

Let us do another example. Suppose we have a long solenoid, with many turns. We have previously discussed that it generates a uniform $\boldsymbol{B}$ inside, and $\boldsymbol{B} = 0$ outside. It turns out that $\boldsymbol{A} \neq 0$ outside of the solenoid, so let us calculate what it is. Let us impose the condition that it should be invariant moving upwards or around the solenoid (based on the symmetry of the problem), in the z or ϕ directions. It should only be dependent on r. We generate a circle around the solenoid, with radius r, and compute the line integral of $\boldsymbol{A}$:

$$\oint \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{l}$$

We can use Stoke's theorem to write this as a surface integral, where we note that the surface is not closed, so we cannot say it is 0 through Maxwell' third equation:

$$\oint \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{l} = \iint \nabla \times \boldsymbol{B} \cdot d\boldsymbol{s}$$

This is called the magnetic flux.

We know that inside the solenoid, we have that $\boldsymbol{B} = \mu_0 n I \hat{z}$. We can therefore calculate the magnetic flux inside the solenoid:

$$A_\phi 2\pi r = \mu_0 n I \pi R^2$$

Where R is the radius of the solenoid. Therefore, the A field outside of the solenoid is given by

$$A_\phi = \frac{\mu_0 n I R^2}{2r}$$

And if we have the case where our surface is inside the solenoid, we find that

$$A_\phi^{\text{inside}} = \frac{\mu_0 n I}{2} r$$

Suppose we now introduce a charged particle moving outside of the solenoid. This particle cannot see the $\boldsymbol{B}$ field, but its wavefunction gains a phase due to the $\boldsymbol{A}$ field:

$$\exp\left(iq \oint \boldsymbol{A} \cdot d\boldsymbol{l}\right)$$

This means that if we have a particle going halfway around the loop, and another going halfway in the other direction, they will differ by a phase. This will generate an interference pattern. This is known as the Aharanov-Bohm effect. The phase must also be quantized, due to quantum mechanics, and thus leads to a fundamental unit for the magnetic flux, known as the flux quanta.

Suppose we have a solenoid of radius R, with n turns per unit length, and current I. We know that inside the solenoid, $\boldsymbol{B} = \mu_0 n I \hat{z}$, and outside, $\boldsymbol{B} = 0$. We want to find $\boldsymbol{A}$. We can use a modification of Ampere's Law:

$$\int_{\mathcal{S}} d\boldsymbol{s} \cdot \boldsymbol{B} = \Phi_B(\mathcal{S})$$

Now applying Stoke's theorem:

$$\int_{\mathcal{S}} d\boldsymbol{s} \cdot (\nabla \times \boldsymbol{A}) = \Phi_B(\mathcal{S})$$

$$\oint_{\partial \mathcal{S}} d\boldsymbol{r} \cdot \boldsymbol{A} = \Phi_B(\mathcal{S})$$

We know that by the condition that the curl must be in the $\hat{z}$ direction, we need that $\boldsymbol{A} = |\boldsymbol{A}|\hat{\phi}$. When we are inside the solenoid, $r < R$, the left side integral is $2\pi r|\boldsymbol{A}|$, and the right side will be $\mu_0 I n \pi r^2$. From this:

$$\boldsymbol{A} = \frac{\mu_0 n I}{2}\hat{\phi}$$

If we have that $r > R$, we have

$$2\pi r|\boldsymbol{A}| = \mu_0 n I \pi R^2$$

Where the flux is 0 everywhere outside the solenoid, so we just have the area of the solenoid. From this:

$$\boldsymbol{A} = \frac{\mu_0 I n R^2}{2r}\hat{\phi}$$

Suppose we have two wires with line charge density λ, separated by a distance d. If the currents are moving in the same direction, what does the speed of the electrons have to be for the forces to be balanced? The electric field magnitude is given by

$$|\boldsymbol{E}| = \frac{\lambda}{2\pi\epsilon_0 d}$$

The force per unit length will be

$$f_e = \frac{\lambda^2}{2\pi\epsilon_0 d}$$

The magnetic field from a wire is given by

$$|\boldsymbol{B}| = \frac{\mu_0 I}{2\pi d}$$

And the force per unit length is given by

$$f_m = \frac{\mu_0 I_1 I_2}{2\pi d}$$

In this case, the current will be $\boldsymbol{I} = \lambda \boldsymbol{v}$, and thus

$$f_m = \frac{\mu_0 \lambda^2 \boldsymbol{v}^2}{2\pi d}$$

For these two forces to balance, we have that

$$f_e = f_m \to \frac{\lambda^2}{2\pi\epsilon_0 d} = \frac{\mu_0 \lambda^2 v^2}{2\pi d} \to v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

This is the speed of light.

2.7 Multipole expansion

Suppose we have a current distribution, and we want to calculate the vector potential far away, at a point $\boldsymbol{r}$. We have that

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r'})\, d^3\boldsymbol{r'}}{R}$$

We can do a Taylor expansion of $\frac{1}{R}$:

$$\frac{1}{R} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos\theta)$$

Where θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r'}$. The term where $l = 0$ is known as a monopole, and $l = 1$ gives us the dipole, $l = 2$ gives the quadropole, etc.

Let us begin with $l = 0$. In this case:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi r} \int \boldsymbol{J}\, d^3\boldsymbol{r'}$$

When we restrict ourselves to physical reality, then we must have that this integral is 0, and thus the magnetic monopole term is 0. Thus, we have no magnetic monopoles.

We can then move to the case where $l = 1$. We now have that

$$\frac{r'}{r^2} P_1(\cos\theta) = \frac{r'}{r^2}\cos\theta = \frac{1}{r^2}\boldsymbol{r'} \cdot \hat{r} = \frac{1}{r^3}\boldsymbol{r'} \cdot \boldsymbol{r}$$

Inserting this into $\boldsymbol{A}$:

$$\boldsymbol{A} = \frac{\mu_0}{4\pi r^3} \int \boldsymbol{r} \cdot \boldsymbol{r'} \boldsymbol{J}(\boldsymbol{r'}) d^3\boldsymbol{r'}$$

We can do this all out, and we will find that

$$\boldsymbol{A} = -\frac{\mu_0}{8\pi r^3} \boldsymbol{r} \times \int \boldsymbol{r'} \times \boldsymbol{J}(\boldsymbol{r'})\, d^3\boldsymbol{r'}$$

We then define the magnetic moment:

$$\boldsymbol{m} = \frac{1}{2} \int \boldsymbol{r'} \times \boldsymbol{J}(\boldsymbol{r'})\, d^3\boldsymbol{r'}$$

Thus we can rewrite $\boldsymbol{A}$:

$$\boldsymbol{A} = \frac{\mu_0}{r\pi r^3}(\boldsymbol{m} \times \boldsymbol{r})$$

Let us now try to understand what this means. Let us imagine that we have a current loop of radius R, with constant current flowing around. Let the center of the loop be the origin. In this case, $\boldsymbol{J}(\boldsymbol{r'}) = I\, d\boldsymbol{l}$. Thus the inside of the integral has $\boldsymbol{r} \times d\boldsymbol{l}$. This is twice the area of the wedge traced out, and thus the overall integral will be twice the area of the circle, times the constant current since we can pull that out:

$$\boldsymbol{m} = \frac{1}{2} \int \boldsymbol{r'} \times \boldsymbol{J}(\boldsymbol{r'})\, d^3\boldsymbol{r'} = \frac{1}{2} \int \boldsymbol{r} \times I\, d\boldsymbol{l} = AI\hat{z}$$

We can think of the magnetic moment as the area times the current.

Using this magnetic moment for the current loop, we can compute the $\boldsymbol{B}$ field:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{\mu_0}{4\pi r^3}\left[3\boldsymbol{m} \cdot \hat{r}\hat{r} - \boldsymbol{m}\right] + \frac{2\mu_0}{3}\boldsymbol{m}\delta^3(\boldsymbol{r})$$

3 Bound Currents and Magnetism

There are two types of charges, free charges, such as charges that flow through wires, or live on the plates of capacitors, generally associated with metals, and bound charges, those that are confined to atoms, and are hard to move around. These bound charges are associated with insulators, and can be polarized to generate dipole moments, which generate dipole fields.

We have the same duality for currents, where we have free currents, which are the currents that flow through electric lines and wires, and we have bound currents, currents that are confined, such as those inside atoms. In fact, all of magnetism is generated from these bound currents inside materials. We will talk about the different manifestations of magnetism.

All of magnetism starts with atoms. Let us take the simplest atom, Hydrogen. We have a proton, with positive charge, and an orbiting electron, with negative charge. This forms a current loop, and generates a magnetic field. This generates a dipole moment, which we can approximate using a fully classical model, which we will later patch up with quantum mechanics. We have circular motion with distance r, and assume that the electron moves around with some velocity v. We want to compute the magnetic dipole moment of this:

$$\boldsymbol{m} = I\boldsymbol{a}$$

The current is defined in two ways, the charge times the velocity, as well as the rate at which charge flows past a certain point, which is the definition that we will use:

$$I = e\frac{v}{2\pi r}$$

And we know that the area is given by πr^2, and thus the magnitude of $\boldsymbol{m}$ is

$$\boldsymbol{m} = \pi r^2 e\frac{v}{2\pi r} = \frac{1}{2}erv$$

We can write this in terms of the orbital angular momentum:

$$\ell_z = r \times p = rmv \to |\boldsymbol{m}| = \frac{e}{2m}\ell_z$$

We can relate this to the Bohr magneton, which has magnetic moment:

$$\mu_b = \frac{e\hbar}{2mc}$$

We can write our magnetic moment in terms of the Bohr magneton moment:

$$|\boldsymbol{m}| = \mu_0\frac{\ell_z}{\hbar}$$

However, we have neglected the spin of the electron. Each electron is like a spinning current loop, and thus it will generate its own magnetic moment, which we add onto what we have for the atom:

$$|\boldsymbol{m}| = \mu_b\ell_z + 2\mu_B\boldsymbol{s}$$

Where we have dropped the $\hbar$ in the denominator of the first term because angular momentum is quantized in terms of $\hbar$, and the 2 is there because the eigenvalues of the spin operator (related to Pauli matrices) are $\pm\frac{1}{2}$, and thus we need to cancel it out.

It turns out that in truth the 2 is not actually just 2, it is 2 plus something, which is related to QED. This extra term is due to J. Schwinger, who shared a Nobel Prize with Feynman for his development of QED. This "something" turned out to be $\frac{\alpha}{2\pi}$, where α is the fine structure constant.

We can look at the total magnetization $\boldsymbol{M}$, which is nonzero when we have a magnet, when the alignment of the spins are not cancelling out.

Suppose we take a piece of material in an external magnetic field $\boldsymbol{B}$, where the original material has $\boldsymbol{M} = 0$. The total force on each of the atoms is 0, but the magnetic field induces a torque:

$$\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B}$$

Let us first derive this induced torque. We have a rectangular current loop in the xy plane, which has $\boldsymbol{m}$ pointing upwards, with current flowing clockwise. We insert a $\boldsymbol{B}$ field towards the right. The left and right sides have side length a, and the forward and backward edges have side length b.

The force is equal to the current crossed with the $\boldsymbol{B}$ field:

$$\boldsymbol{F} = \boldsymbol{I} \times \boldsymbol{B}$$

Thus only the current segments that are orthogonal will lead to a force, which are only the left and right segments. The force on the left side is $|\boldsymbol{F}_{left}| = IaB$, facing downwards. On the right side, we also have magnitude IaB, but facing upwards, since the current flow is in the opposite direction. Thus there is no net force on the current loop.

We can compute the torque, which is $\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}$. We see that there will only be torques on the left and right edges, and the torque will be of magnitude $\frac{b}{2}IaB$ for both, this time in the same direction. Thus we have that

$$|\boldsymbol{N}| = (abI)B = |m|B$$

Looking at directions, this is actually $\boldsymbol{B} \times \boldsymbol{m}$. So we have derived that the torque on the magnet will be $\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{m}$.

How do we explain the forces between two magnets? The reason for this is that the $\boldsymbol{B}$ field is not uniform, and thus the forces will not cancel out. It can be shown that if we put a piece of magnet in an external $\boldsymbol{B}$ field, it feels a torque, but also has an energy, $U = -\boldsymbol{m} \cdot \boldsymbol{B}$. From this, we want to take the gradient to get the force, $\boldsymbol{F} = -\nabla U = \nabla(\boldsymbol{m} \cdot \boldsymbol{B})$. Thus we see that we can exert a force if we have a non-uniform $\boldsymbol{B}$ field. This is known as the Zeeman effect, and he got a Nobel Prize for this.

This is very similar to the dipole from electrostatics, where we have the Stark effect:

$$U = -\boldsymbol{p} \cdot \boldsymbol{E}$$

We also had a torque:

$$\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E}$$

And we have the exactly analogous:

$$\boldsymbol{F} = -\nabla U = \nabla(\boldsymbol{p} \cdot \boldsymbol{E})$$

3.1 Paramagnetism

Suppose we have a material that has $\boldsymbol{M} = 0$, and we place it in a $\boldsymbol{B}$ field, that then changes the magnetization to $\boldsymbol{M} = \chi_m \boldsymbol{B}_{\text{ext}}$, where χ_m is called the magnetic susceptibility. This is how a magnet picks up nails, it magnetizes the nails, and then produces an attractive force. The external $\boldsymbol{B}$ field induces a magnetic dipole field. Note that in the textbook, the external field is denoted as $\boldsymbol{H}$.

If we write down the equation for Ampere's Law:

$$\nabla \times \boldsymbol{B} = \mu_0(\boldsymbol{J}_{free} + \boldsymbol{J}_{bound})$$

It turns out that $\boldsymbol{J}_{bound} = \nabla \times \boldsymbol{M}$:

$$\nabla \times (\boldsymbol{B}_{total} - \boldsymbol{M}\mu_0) = \mu_0 \boldsymbol{J}_{free}$$

$$\to \nabla \times \left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right) = \boldsymbol{J}_{free}$$

This inner term is known as $\boldsymbol{H}$:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{free}$$

3.2 Ferromagnetism

Suppose we have a piece of material that has a bunch of domains, local regions of aligned magnetic moments. However, these domains are still macroscopically oriented in a way so that the net magnetization is still 0. When we apply an external field, certain domains grow, and certain domains shrink, but we don't get a fully polarization. We have that

$$\boldsymbol{M} \propto \boldsymbol{B}_{ext} \propto \boldsymbol{H}$$

It turns out that if we plot the curve of the total magnetization, we have a hysteresis curve.

3.3 Diamagnetism

The third kind of magnetic material is composed of atoms that have no magnetic moment to start with, thus $\boldsymbol{M} = 0$. When we place the material into a magnetic field, it will induce a magnetic moment in the opposite direction of the external field. The reason for this is that we can model an atom as an electron orbiting the nucleus. The Coulomb force provides the centripetal acceleration. When we introduce a magnetic force, we add in the Lorentz force, $q\boldsymbol{v} \times \boldsymbol{B} = evB$, which will change the velocity of the orbiting electron:

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2} \to \frac{mv^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2} + eBv$$

It turns out that the change in magnetic moment is opposite the direction of the $\boldsymbol{B}$ field, which is known as diamagnetism.

3.4 Bound and Free Currents

We can compute the vector potential due to a material with some magnetization density $\boldsymbol{M}$:

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{M}(\boldsymbol{r'}) \times \hat{R}}{R^2}\, d^3\boldsymbol{r'}$$

We can use the fact that

$$\frac{\boldsymbol{R}}{R^2} = \nabla_{\boldsymbol{r}}\frac{1}{R} = -\nabla_{\boldsymbol{r'}}\frac{1}{R}$$

And rewrite the inside of the integral:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \boldsymbol{M}(\boldsymbol{r'}) \times -\nabla_{\boldsymbol{r'}}\frac{1}{R}\, d^3\boldsymbol{r'}$$

We can write this inside portion as

$$\boldsymbol{M} \times \left(-\nabla\frac{1}{R}\right) = -\nabla_{\boldsymbol{r'}} \times \frac{\boldsymbol{M}}{R} + \frac{\nabla \times \boldsymbol{M}}{R}$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int \frac{\nabla \times \boldsymbol{M}}{R}\, d^3\boldsymbol{r'} - \frac{\mu_0}{4\pi}\int \nabla_{\boldsymbol{r'}} \times \frac{\boldsymbol{M}}{R}\, d^3\boldsymbol{r'}$$

We note that the first term is of the same form as the integral for the magnetic vector potential due to a current:

$$\boldsymbol{J}_{\text{bound}} = \nabla \times \boldsymbol{M}(\boldsymbol{r})$$

The second term is a surface current density, $\boldsymbol{K}(\boldsymbol{r}) = \boldsymbol{M}(\boldsymbol{r}) \times \boldsymbol{n}_s$, where $\boldsymbol{n}_s$ is the surface normal, leaving the magnetic vector potential as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}_b}{R}\, d^3\boldsymbol{r'} + \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{K}}{R}\, ds$$

Intuitively, when $\boldsymbol{M}$ is uniform, we expect the body current density to be 0, because each atom is a current loop, and the current loops next to each other will have currents going in opposite directions, leading to cancellation. The only currents that will not cancel will be the current on the outside, giving the effective result of an electron going around the material.

What happens if the magnetization is not uniform? Let us assume that the magnetization is increasing along the y direction, we have larger and larger current loops with magnetization pointing in the z directions, as we go along the y direction. We intuitively still have cancellation, but the larger current loops overpower the smaller currents. Thus we have a current in the x direction:

$$J_x \sim \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} = (\nabla \times \boldsymbol{M})_x$$

Let us do an example of a calculation of a magnetic field. Suppose we have a sphere with uniform magnetization along the z direction, $\boldsymbol{M} = M\hat{z}$. We want to calculate the $\boldsymbol{B}$ field. We do this via calculating $\boldsymbol{A}$:

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{M} \times \hat{R}}{R^2}\, d^3\boldsymbol{r'}$$

However, we could also compute this by saying that there is no body current, $\boldsymbol{J}_b = \nabla \times \boldsymbol{M} = 0$. We can then compute $\boldsymbol{K} = \boldsymbol{M} \times \boldsymbol{n}$. In this case, $\boldsymbol{n} = \hat{r}$, and thus we have that

$$\boldsymbol{K} = M\sin\theta\hat{\phi}$$

This is the same as a uniform surface charge density on a sphere, rotating at constant ω. The answer for $\boldsymbol{A}$ for this we have seen in the textbook. We have that $\sigma\omega R = M$, because in that problem we have $\boldsymbol{K} = \sigma\omega R\sin\theta\hat{\phi}$. Using the solution of that problem, we have that inside the sphere, we have a constant field:

$$\boldsymbol{B}_{\text{inside}} = \frac{2}{3}\mu_0\boldsymbol{M}$$

And outside, we have the magnetic moment given by

$$\boldsymbol{m} = \frac{4}{3}\pi r^3\boldsymbol{M}$$

Recall that Ampere's Law states that

$$\nabla\times\boldsymbol{B} = \mu_0(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}})$$

We know that the bound current is given by $\nabla\times\boldsymbol{M}$:

$$\nabla\times\boldsymbol{B} = \mu_0(\boldsymbol{J}_f + \nabla\times\boldsymbol{M})$$

We can rewrite this:

$$\boldsymbol{J}_f = \nabla\times\left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right)$$

This term is known as $\boldsymbol{H}$,

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}$$

This represents the free current. $\boldsymbol{B}$ governs the total generated magnetic field, and $\boldsymbol{H}$ is the field generated by the free current. In physical reality, we can change $\boldsymbol{H}$, we cannot control $\boldsymbol{B}$. This is why $\boldsymbol{H}$ is oftentimes called the magnetic field.

Suppose we have a current flowing through a magnetic material, in the shape of a cylinder of radius R. The material has some unknown $\boldsymbol{M}$, and we want to compute $\boldsymbol{H}$. We can compute this via Ampere's Law, disregarding the material properties. Using cylindrical coordinates, we are some r away from the center (perpendicularly):

$$\oint\boldsymbol{H}\cdot d\boldsymbol{l} = I_{total}\mu_0$$

We make a contour, a circle with radius r:

$$2\pi rH = \frac{I}{\pi R^2}\pi r^2 = \frac{Ir^2}{R^2}$$

Where the right side is the current per unit area for the cylinder, times the area of the contour. Thus we have that

$$H = \frac{Ir}{2\pi R^2}$$

If we take a contour outside, we have that

$$2\pi rH = I$$

And thus we have that

$$H = \frac{I}{2\pi r}$$

If we really wanted $\boldsymbol{B}$, we would need an equation of state. We have 3 types of materials, as discussed before. For a paramagnet, we have that

$$\boldsymbol{M} = \chi_m \boldsymbol{H}$$

Thus we have that

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} = \frac{\boldsymbol{B}}{\mu_0} - \chi_m \boldsymbol{H}$$

This gets us that

$$\boldsymbol{B} = \mu_0(1+\chi_m)\boldsymbol{H} = \boldsymbol{B}$$

The term $1+\chi_m$ is called the relative permeability, μ_r, and together we have that $\mu_0\mu_r = \mu$, the permeability:

$$\mu_0\mu_r\boldsymbol{H} = \mu\boldsymbol{H} = \boldsymbol{B}$$

If we go back to our example, if the material has some permeability μ, we have that

$$\boldsymbol{B}_{\text{inside}} = \frac{Ir\mu}{2\pi R^2}$$

$$\boldsymbol{B}_{\text{outside}} = \frac{\mu_0 I}{2\pi r}$$

We note that the $\boldsymbol{B}$ field is discontinuous across the surface of the material. Recalling the boundary conditions from Maxwell's equations:

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \mu \boldsymbol{J}$$

The first equation tells us that perpendicular $\boldsymbol{B}$ fields are the same above and below. The second equation is telling us that

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \iint \boldsymbol{K} \cdot d\boldsymbol{s}$$

From this, we will find a discontinuity that is given by $\hat{n} \times \boldsymbol{K}$.

Going back to our cylinder, the inside magnetic field will give

$$\boldsymbol{B} = \frac{\mu I}{2\pi R}$$

And the outside field will give

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi R}$$

We see that we have a bigger field inside:

$$\boldsymbol{B}^{||}_{\text{inside}} - \boldsymbol{B}^{||}_{\text{outside}} = \frac{(\mu-\mu_0)I}{2\pi R} = \frac{\mu_0\chi_m I}{2\pi R} = \mu_0\chi_m\boldsymbol{H}^{||} = (\hat{n}\cdot\boldsymbol{K})_\phi$$

This is the boundary condition, the B field is equal to the surface current.

Looking at χ_m, most materials have χ_m that is very small, magnetic effects are quite small compared to electric effects, they are generally around the order of 10^{-5}. In the case of diamagnetism, we haev no magnetism to begin with, we induce a magnetic effect. Landau first described diamagnetism, and he modelled it as the external $\boldsymbol{B}$ field causing the electron orbits to be adjusted, due to the added Lorentz force. This causes an induced magnetic moment, and we model χ as

$$\chi_m = \frac{e^2 r^2}{4m_e}$$

This is Landau diamagnetism.

3.5 Conductivity and Resistivity

We can write the current density in terms of the force per unit charge $\boldsymbol{f}$:

$$\boldsymbol{J} = \sigma \boldsymbol{f}$$

where σ is the conductivity. Combining this with the Lorentz force:

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Now using the fact that the drift velocity of electrons in a wire is almost 0, $|\boldsymbol{v}| \approx 0$, we have that

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

We can also write ρ, the resistivity, which is $\frac{1}{\sigma}$. Taking a look at some substances and their resistivities, we have that silver, Ag, has a resistivity of $\rho = 1.59 \times 10^{-8}$. We also see that copper, Cu, has a resistivity of 1.68×10^{-8}, which is why we use it for wires. For high resistivities, we have that pure water has a resistivity of 2.5×10^5, and glass has a resistivity on the order of 10^{10}. Note that if we have a wire, the resistance is given by

$$R = \frac{\rho L}{A}$$

Where L is the length and A is the cross-sectional area.

Suppose we have some material with cross sectional area A and length L, with some electric field put across it, given by some potential different V. The current will be

$$I = JA = \sigma \boldsymbol{E} A = \frac{\sigma V A}{L}$$

Solving that for V:

$$V = I\frac{L}{\sigma A} = \frac{\rho L}{A} I = IR$$

Thus we have Ohm's Law.

Let us look at coaxial cylinders, as a slightly less simple example. We have two concentric cylinders, with length L, and the inner one has radius a and the outer one has radius b. Between them, there is a material with conductivity σ. If we put a potential difference across the two, what is the current that flows between them?

The electric field is given by

$$\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

Where λ is the line charge density, and $\hat{r}$ points radially outwards. The total current will be given by

$$I = \int \boldsymbol{J} \cdot d\boldsymbol{s} = \sigma \int \boldsymbol{E} \cdot d\boldsymbol{s} = \sigma \int_0^L dz \int_0^{2\pi} d\phi r \frac{\lambda}{2\pi\epsilon_0 r} = \frac{\sigma\lambda L}{\epsilon_0}$$

The potential difference is given by

$$V = -\int_b^a d\boldsymbol{l} \cdot \boldsymbol{E} = -\int_b^a \frac{\lambda}{2\pi\epsilon_0 r}\hat{r} \cdot \hat{r} = \frac{\lambda}{2\pi\epsilon_0} \ln\frac{b}{a}$$

From this, we can see the relationship:

$$I = \frac{2\pi\sigma L}{\ln\frac{b}{a}}V$$

The Drude Model is a model of a wire that is full of electrons, and as an electron move along, it strikes other electrons. The average distance that an electron moves before it strikes something is called the mean free path. This model gives us that the voltage is the work per unit charge, and that current is the charge per unit time. From these, we can find the power, the work per unit time, as $P = IV = I^2R = \frac{V^2}{R}$.

3.6 Electromotive Force

Suppose we have a circuit, with a battery or some power source, connected to a light bulb. When we switch the power on, why is there current? Why don't some of the electrons just move out of the positive terminal and some electrons come in through the negative terminal? And since the drift velocity is so slow for electrons in a wire, why does it not take forever for the light bulb to turn on?

We can prove this via contradiction. Let us assume that the current is not uniform. If we zoom in on a section of wire, and the current was not uniform, then charge must be piling up somewhere. We would have a net charge, and thus have an induced electric field. Because of this, the charges moving in and out of the wire segment would start to even out. This would reduce the electric field, and thus we have a relf-regulating stability, at even current everywhere. Thus we have that $\boldsymbol{f}$ is comprised of two forces (measured in force per unit charge here to make units work out), the power source force, and the electric field, which maintains current uniformity:

$$\boldsymbol{f} = \boldsymbol{f}_s + \boldsymbol{E}$$

We then define the electromotive force:

$$\varepsilon = \oint_{\text{circuit}} \boldsymbol{f}\cdot d\boldsymbol{r}$$

Note that if we compute this for electrostatics, where $\nabla\times\boldsymbol{E} = 0$, we find that $\varepsilon = \oint_{\text{circuit}} \boldsymbol{f}_s\cdot d\boldsymbol{r}$.

Suppose we have two regions of space. In one region, we have a magnetic field going into the page. We have a loop of wire that is part inside and part outside the field. We have that the loop is closed by some load. The loop is a distance x into the field area (as we pull the loop out of the magnetic field area, x decreases). The loop has height h, and thus hx is the area of the loop inside the $\boldsymbol{B}$ field. If we pull the loop out with some velocity $\boldsymbol{v}$, what happens?

In this case, the Lorentz force is given by

$$\boldsymbol{f} = \boldsymbol{v}\times\boldsymbol{B}$$

And the electromotive force is given by

$$\varepsilon = \oint \boldsymbol{f}_{mag}\cdot d\boldsymbol{r}$$

In this case, the force $\boldsymbol{f}$ will point upward. When we integrate along the wire, the horizontal parts of the loop will not contribute, and thus the only contribution will be the upwards segment of length h:

$$\varepsilon = \int vB\,dr = vBh$$

We can talk about the magnetic flux:

$$\Phi_B = Bhx$$

If we look at the change in the flux:

$$\frac{d\Phi_B}{dt} = Bh\frac{dx}{dt} = -Bhv$$

We have shown that $\varepsilon = -\frac{d\Phi_B}{dt}$.

Suppose we have a current loop that changes shape, from time t to time dt. We will have some "ribbon" of area that is new/changed, and thus the change in flux will be

$$\Phi_B(t+dt) - \Phi_B(t) = d\Phi_B = \int_{\text{ribbon}} \boldsymbol{B} \cdot d\boldsymbol{A}$$

We can take infinitesimal square segments of the ribbon, and we have a $d\boldsymbol{A}$ that is perpendicular to the ribbon surface, and is given by $d\boldsymbol{A} = \boldsymbol{v}dt \times d\boldsymbol{r} = (\boldsymbol{v} \times d\boldsymbol{r})dt$. Thus we have that the change in flux, or the flux through the ribbon is given by

$$\int_{\text{ribbon}} \boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{r})dt = \int_{\text{ribbon}} d\boldsymbol{r} \cdot (\boldsymbol{B} \times \boldsymbol{v})dt = -\int_{\text{ribbon}} d\boldsymbol{r} \cdot (\boldsymbol{v} \times \boldsymbol{B})\, dt$$

And from this we have that

$$\frac{d\Phi_B}{dt} = -\int d\boldsymbol{r}(\boldsymbol{v} \times \boldsymbol{B}) = -\oint d\boldsymbol{r} \cdot \boldsymbol{f}_{mag} = -\varepsilon$$

Thus we have shown that in the general case:

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

This will soon come in handy when dealing with Maxwell's Third Equation, in its full form.

Suppose that we have a rotating disk in a magnetic field, with radius a, and rotating with angular velocity ω. The magnetic field points up along the axis, and the disk is in contact with two circuit leads, one at the center of the disk and one at the rim. There is a resistor with resistance R hooked up between the leads. We want to find the current in the resistor. The emf is given by

$$\varepsilon = \oint \boldsymbol{f}_{mag} \cdot d\boldsymbol{r} = \int vB\, dr = \int \omega r B\, dr = \frac{\omega B a^2}{2}$$

The current through the resistor will be

$$I = \frac{\varepsilon}{R} = \frac{\omega B a^2}{2R}$$

Going back to the loop moving in and out of the magnetic field, suppose we instead move the magnetic field, rather than the loop of the circuit. Intuitively, we should have an induced emf, but the charges have no velocity, so there is no Lorentz force, so how can there be an emf? Faraday deduced that there must be an electric field, that is the only thing that could cause a static charge to move:

$$\varepsilon \to \oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}$$

Where $\Phi = \oiint_S \boldsymbol{B} \cdot d\boldsymbol{s}$. Faraday then set up a third experiment, where he had the same setup, but had both the magnetic field and the circuit static, but let the magnetic field vary in strength, $\boldsymbol{B}(t)$. Due to this, the flux is changing, and we have a generated emf, with everything static. Thus we have that

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt} \iint \boldsymbol{B} \cdot d\boldsymbol{s} = -\iint \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s}$$

Rewriting the left side:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = \iint \nabla \times \boldsymbol{E} \cdot d\boldsymbol{s}$$

And thus we have that

$$\iint \nabla \times \boldsymbol{E} \cdot d\boldsymbol{s} = -\iint \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s}$$

From this, we have that

$$\boxed{\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}}$$

In electrostatics, this curl of the electric field was 0, but when we have a changing magnetic field, we have a contribution from $\boldsymbol{B}$. This is the equation that connects electric fields and magnetic fields. If we have that $\nabla \times \boldsymbol{E} = 0$, and $\boldsymbol{E} = -\nabla V$, this means that the potential is only valid when we have uniform magnetic fields. However, we can salvage this, by writing down an equation for $\boldsymbol{E}$ that explicitly looks at the potential segment and the magnetic segment. We now have that

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Now we recall that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Inserting this:

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t} \nabla \times \boldsymbol{A} = -\nabla \times \frac{\partial \boldsymbol{A}}{\partial t}$$

Now we can cancel out the curls (mathematicians beware!):

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}$$

And now we can write out the total electric field:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla V$$

It will turn out that this is consistent with Einstein's special relativity. Note that we now have to put constraints on V for the gauge transformation:

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla \Phi$$

$$V \to V + \frac{\partial \Phi}{\partial t}$$

This is the complete gauge transformation.

Let us now look at examples. Suppose we have a circular disc, with a magnetic field $\boldsymbol{B}(t)$ pointing upwards. We want to find the electric field generated by this changing magnetic field.

By symmetry, the $\boldsymbol{E}$ field an equal distance from the center of the disc will be equal in magnitude. We can compute the emf at a certain radius r:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = E2\pi r = -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t}\left[\pi r^2 B(t)\right] = -\pi r^2 \frac{\partial B}{\partial t}$$

Thus we have that

$$E = -\frac{1}{2} r \frac{\partial B}{\partial t}$$

What does the sign here mean? There is a law, known as Lenz's Law, which states that the induced effect will always resist the change that induced it. If we let $\boldsymbol{B}$ increase in time, a charge in the disc will generate a current, which will try to make $\boldsymbol{B}$ smaller, meaning that it will flow clockwise. On the other hand, if $\boldsymbol{B}$ is decreasing, then the current will flow counterclockwise.

3.7 Inductance

We can now start talking about inductance. If we have a circuit, and some distance away we have a second circuit, and we produce a current in the first circuit, I_1, it will generate a magnetic field $\boldsymbol{B}$, which will have some flux through the circuit $\Phi_2 = M_{21} I_1$, where M_{21} is the mutual inductance. This is a physical property of the setup, based on the two circuits and the distance between them. Similarly, we can run a current through the second circuit, and we have $\Phi_1 = M_{12} I_2$, and we have that $M_{12} = M_{21}$.

This is how transformers work, where the two circuits have different sizes and number of turns per coil. Suppose we have a coil with radius a, length l_1, and a turn density of n_1. We have a concentric larger coil, of radius b, length l_2, and a turn density of n_2. We can compute the magnetic field between the two coils using Ampere's Law:

$$\mu_0 n_2 l I = \oint \boldsymbol{B} \cdot d\boldsymbol{l} = Bl \to B = \mu_0 n_2 I$$

Now we can compute flux through the inner coil:

$$\pi a^2 \, \mu_0 n_2 I \, n_1 l_1 = M_{12} I$$

$$M_{12} = \pi a^2 n_1 n_2 l_1 \mu_0$$

Let us now talk about self-inductance. Suppose we have a coil, with some current I running through it. It will have some flux, which is proportional to the current:

$$\Phi = LI$$

Where L is known as the self-inductance. This is measured in Henries.

Let us look at a single coil, with some current I flowing through it, the inductance will resist this, because we know that

$$\varepsilon = -\frac{\partial \Phi}{\partial t} = -L\frac{dI}{dt}$$

The work that we do, is equal to the change in the emf:

$$dW = I \, d\varepsilon = IL\frac{dI}{dt}$$

This is the work that we have to do to change the current in the circuit. Suppose we start from 0 current, and end up with a current I in the circuit. This takes:

$$W = \int_0^\infty LI\frac{dI}{dt}\,dt = \frac{1}{2}LI_f^2$$

Therefore, we have that inductors store energy. Now we note that $LI = \Phi$:

$$W = \frac{1}{2}\Phi I = \frac{1}{2}I\iint \boldsymbol{B}\cdot d\boldsymbol{s}$$

$$= \frac{1}{2}I\iint \nabla\times\boldsymbol{A}\,d\boldsymbol{s}$$

Now using Stoke's theorem:

$$W = \frac{1}{2}I\int \boldsymbol{A}\cdot d\boldsymbol{l} = \frac{1}{2}\int I\boldsymbol{A}\cdot d\boldsymbol{l} = \frac{1}{2}\int \boldsymbol{A}\cdot\boldsymbol{J}\,d^3\boldsymbol{r}$$

Where we have used the fact that $I\cdot d\boldsymbol{l} = \boldsymbol{J}$. Thus we have that

$$W = \frac{1}{2}\int \boldsymbol{A}\cdot\boldsymbol{J}\,d^3\boldsymbol{r}$$

We can now change this even further, by noting that this is

$$W = \frac{1}{2\mu_0}\int \boldsymbol{A}\cdot\nabla\times\boldsymbol{B}\,d^3\boldsymbol{r}$$

Now using vector identities:

$$W = \frac{1}{2\mu_0}\int \nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})\,d^3\boldsymbol{r} + \frac{1}{2\mu_0}\int(\nabla\times\boldsymbol{A})\cdot\boldsymbol{B}\,d^3\boldsymbol{r}$$

Now if we let this left integral be over all space, the integral is 0, and thus we have

$$W = \frac{1}{2\mu_0}\int \boldsymbol{B}^2 d^3\boldsymbol{r}$$

This is analogous to the electric

$$E = \frac{\epsilon_0}{2}\int \boldsymbol{E}^2\,d^3\boldsymbol{r}$$

4 Electrodynamics

We have two of Maxwell's equation:

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla\cdot\boldsymbol{B} = 0$$

These two equations have no time dependence. We also have the equation

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$

Which has time dependence. We also have that

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}$$

Which has no time dependence. Maxwell looked at these, and discovered that there was an inconsistency in these equations. We begin with current conservation, the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

Now inserting the equations that we have:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = \frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \boldsymbol{E}) + \frac{1}{\mu_0}\nabla \cdot (\nabla \times \boldsymbol{B})$$

The second term is 0 (by vector identities), but in general, the first term is not 0:

$$\epsilon_0 \nabla \cdot \frac{\partial E}{\partial t} \neq 0$$

This violates the continuity equation! We expect this to be equal to 0!

One can argue that this is just mathematics, so let us look at a physical situation where we have an inconsistency. Suppose we have a parallel plate capacitor in a circuit with a battery, that is charging the capacitor, with a switch that completes the circuit. At $t = 0$, let us assume that the switch is on, and therefore we have a current I flowing through the capacitor. This current flow will be time dependent, because the capacitor is charging up. The charge on the plate, Q, is a function of t, $Q(t)$. Let us now use Maxwell's equations to analyze this situation. Using Ampere's law, we want to find the magnetic field around the wire:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \oiint \boldsymbol{J} \cdot d\boldsymbol{s}$$

Suppose the area is a plate, and the current over the plate is just I:

$$\mu_0 \oiint \boldsymbol{J} \cdot d\boldsymbol{s} = \mu_0 I(t)$$

Suppose we instead choose a surface that contains one plate of the capacitor. Suddenly, we have that the total current flowing through the surface is 0.

How do we solve this problem? We know what the capacitance of the capacitor is, $C = \frac{\epsilon_0 A}{d}$, and as the capacitor is charged, it sets up a potential across the two plates. The potential is given by $V(t) = \frac{Q(t)}{C}$. This is a time dependent potential. Once we have a potential difference, we have an electric field between the capacitors:

$$|\boldsymbol{E}(t)| = \frac{V(t)}{d} = \frac{Q(t)}{Cd} = \frac{Q(t)}{\epsilon_0 A}$$

We see that we have a time dependent electric field. Maxwell then claimed that a time varying electric field, can generate a new type of current, known as the displacement current:

$$\boldsymbol{J}_d = \epsilon_0 \frac{dE}{dt} = \frac{1}{A} I(t)$$

He claimed that Ampere's Law is not complete, and that we must add the displacement current:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \boldsymbol{J}_d = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{d\boldsymbol{E}}{dt}$$

When we add this to our analysis, we see that the displacement current term gives us the $\mu_0 I(t)$ that we were missing when we had our capacitor plate in the surface. Now looking at our four equations, we have a time dependent term for both the magnetic and electric fields. This was the last important addition to Maxwell's equations.

Taking these, along with the Lorentz force ($F = q\boldsymbol{E}+q\boldsymbol{v}\times\boldsymbol{B}$), we can describe all of electromagnetism.

We can rewrite Maxwell's equations in a way that will allow us to more easily derive the wave equation:

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} = 0$$
$$\nabla\times\boldsymbol{B} - \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} = \mu_0\boldsymbol{J}$$

When we have a medium, the medium is characterized by the polarization density $\boldsymbol{P}$, and the magnetization density, $\boldsymbol{M}$. The polarization density will contribute to the bound charge density, $\rho_b = -\nabla\cdot\boldsymbol{P}$. The magnetization density generates a bound current density, $\boldsymbol{J}_b = \nabla\times\boldsymbol{M}$. However, these may change over time, so we have to introduce time dependence to $\boldsymbol{P}$ and $\boldsymbol{M}$. If we introduce time, we introduce an extra current due to the dipole, $\boldsymbol{J}_p = \frac{\partial\boldsymbol{P}}{dt}$. We can write out the total charge:

$$\rho = \rho_f + \rho_b$$

And the total current density

$$\boldsymbol{J} = \boldsymbol{J}_f + \nabla\times\boldsymbol{M} + \frac{\partial\boldsymbol{P}}{\partial t}$$

Once we have this, we can look at Maxwell's equations:

$$\nabla\cdot\boldsymbol{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla\cdot\boldsymbol{P})$$

Now looking at the electric displacement instead:

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$$

$$\nabla\cdot\boldsymbol{D} = \rho_f$$

If we instead look at $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$:

$$\nabla\times\boldsymbol{B} = \mu_0\left(\boldsymbol{J}_f + \nabla\times\boldsymbol{M} + \frac{\partial\boldsymbol{E}}{\partial t}\right) + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{dt}$$

This can be rewritten as

$$\nabla\times\left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right) = \boldsymbol{J}_f + \frac{\partial\boldsymbol{D}}{\partial t}$$

This is the same as

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial\boldsymbol{D}}{\partial t}$$

Thus we have replaced two of the equations with two new ones, that represent the interaction of the medium. However, we don't have a complete set of equations, since we have new variables. We need equations of state, the relationship between $\boldsymbol{B}$ and $\boldsymbol{H}$, and the relationship between $\boldsymbol{D}$ and $\boldsymbol{E}$. In most materials (linear materials), the relationships are given by

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E} \qquad \boldsymbol{M} = \chi_m \boldsymbol{H}$$

Once we have these, we have that:

$$\boldsymbol{D} = \epsilon_0 \epsilon_r \boldsymbol{E} = \epsilon \boldsymbol{E}$$

And similarly for $\boldsymbol{H}$:

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0 \mu_r} = \frac{\boldsymbol{B}}{\mu}$$

We can now solve Maxwell's equations with these two extra relationships.

Let us now do boundary conditions. To do this, we must convert the equations we have to the integral form:

$$\oint\!\!\!\oint \boldsymbol{D} \cdot d\boldsymbol{s} = Q_f$$

$$\oint\!\!\!\oint \boldsymbol{B} \cdot d\boldsymbol{s} = 0$$

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial}{\partial t} \iint \boldsymbol{B} \cdot d\boldsymbol{s}$$

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = I_f + \frac{\partial}{\partial t} \iint \boldsymbol{D} \cdot d\boldsymbol{s}$$

Suppose we are looking at a boundary. When we do the first two equations, we use a matchbox, and we look at the components perpendicular to the surface;

$$D_1^{\perp} - D_2^{\perp} = \sigma_f$$

$$B_1^{\perp} - B_2^{\perp} = 0$$

For the path integrals, we create a rectangular path, and we see that the parallel components are what matter:

$$E_1^{||} - E_2^{||} = 0$$

$$H_1^{||} - H_2^{||} = (\boldsymbol{n} \times \boldsymbol{K})^{||}$$

In a linear medium, we know that $\boldsymbol{D} = \epsilon \boldsymbol{E}$, so we can rewrite the first condition:

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$$

Likewise, we can do something similar with the $\boldsymbol{B}$ condition:

$$\frac{B_1^{||}}{\mu_1} - \frac{B_2^{||}}{\mu_2} = (\boldsymbol{n} \times \boldsymbol{K}_f)^{||}$$

Note that the sign of the currents and the charge are important to keep track of.

Thus we have completed electrodynamics.

4.1 Electromagnetic Energy Density

We have learned that the electric and magnetic fields carry energy:

$$u_{em} = \frac{1}{2}\epsilon_0 \boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2$$

And thus the energy density is given by

$$U_{em} = \int u_{em}\, d^3\boldsymbol{r}$$

Suppose we have some system, and we want to write down the energy conservation relationship for that system:

$$\frac{d(U_{ext} + U_{em})}{dt} + \nabla \cdot \boldsymbol{S} = 0$$

Where U_{ext} is some external energy. $\boldsymbol{S}$ is the energy flow, or energy flux. Recall that the electric energy conservation relationship was

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

We can do out the math, and we find that

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B})$$

This is called the Poynting vector.

Rewriting the conservation relationship:

$$\frac{dU}{dt} = -\oint \boldsymbol{S} \cdot d\boldsymbol{a}$$

Let us consider a circuit, in which we have a piece of cylindrical wire. Let us look at a section of this. This section has some current flowing through it, and has some resistance, which causes a potential difference across the wire section, V. We know that the energy consumption will be $W = VI$. The current flow generates a magnetic field. Suppose the radius of the wire is R, and using Ampere's Law:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I$$

$$2\pi R B = \mu_0 I \to B = \frac{\mu_0 I}{2\pi R}$$

The potential difference will produce an electric field, which must be going from higher potential to lower potential, (let us say that the left side is higher potential), and thus the $\boldsymbol{E}$ field will point from higher to lower, left to right. The magnitude is given by the potential difference V divided by the distance, let us call it l:

$$\boldsymbol{E} = \frac{V}{l}$$

We can now compute the Poynting vector:

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B})$$

Just looking at the right hand rule, we see that the Poynting vector will flow into the wire, at the top, we have a vector pointing down, and at the bottom, we have a vector pointing up.

$$\boldsymbol{S} = \frac{1}{\mu_0}\frac{V}{l}\frac{\mu_0 I}{2\pi R} = \frac{VI}{2\pi Rl}$$

The total energy consumption must be equal to the total area that we have, times the flux:

$$W = 2\pi Rl\frac{VI}{2\pi Rl} = VI$$

We see that this is the same as what we expected. The electromagnetic fields carry the energy, and the resistance of the wire converts this energy, an that is the power that we see in a resistor.

If we have something flowing, we generate a flux, but we also generate a momentum. Thus the energy flow also generates momentum, and so the fields carry momentum. It turns out that the momentum density also has a conservation relationship:

$$\boldsymbol{P}_{em} = \mu_0\epsilon_0\boldsymbol{S}$$

If we want to total momentum, we have to integrate this. This relationship is suspicious, because $\mu_0\epsilon_0 = \frac{1}{c^2}$. This means that we can write the total momentum as

$$\boldsymbol{P} = \frac{1}{c^2}\int \boldsymbol{S}\,d^3\boldsymbol{r}$$

How does momentum conservation work with these electromagnetic fields? We can write down a conservation law:

$$\frac{d(\boldsymbol{P}_{em} + \boldsymbol{P}_{mech})}{dt} + \nabla\cdot \overleftrightarrow{T} = 0$$

Where $\overleftrightarrow{T}$ is known as the stress-energy tensor. The reason we have a tensor is because we can have momentum in a direction and a EM flow in a different direction.

$$T^{ij} = \epsilon_0\left(E^iE^j - \frac{1}{2}\delta^{ij}E^2\right) + \frac{1}{\mu_0}\left(B^iB^j - \frac{1}{2}\delta^{ij}B^2\right)$$

Let us do an example. Suppose we have a coaxial cable, with a potential difference V between them. The outer shell is at negative potential, and the inner shell is positive. We flow current through them (current flowing left to right on the inner shell, and right to left on the outer shell), and we have a load resistor across the two.

If we look at this, we see that we generate an electric field between the two parts, with $\boldsymbol{E}$ pointing in from the inner shell to the outer shell. The current flow generates a magnetic field, which goes out at the top section, and in at the bottom section. At the top section, the Poynting vector will point to the right. We can compute the momentum, so we have to integrate over the surface area, which is the ring between the two shells. If we go through all this, and determine the fields, and then integrate over the surface area, and then multiply by the length of the subsection of the coax cable (because its really a volume integral), we will find that

$$P = \frac{VIl}{c^2}$$

If we work through this using the power through a resistor, we have that $W = VI$, and we know that $P = \frac{W_{em}l}{c^2} = \frac{VI}{c^2}l$, which matches what we computed.

Suppose we now reduce the current to 0. We see that the momentum will disappear, and it turns out that when we do this, we generate a varying electric field, and which generates a magnetic field. This field will push against the cable. If we decrease the current, the momentum in the fields will be transferred into a force acting on the cable.

Let us do an example with the stress-energy tensor. A sphere contains a uniformly distributed charge Q. We can compute the force on the top hemisphere due to the bottom hemisphere. This can be done with the stress-energy tensor. If we look at the momentum:

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{F} = -\nabla \cdot \overset{\leftrightarrow}{T}$$

We can then compute this:

$$\boldsymbol{F} = \int_{\mathcal{V}} \nabla \cdot \overset{\leftrightarrow}{T}\ dV = -\oint \overset{\leftrightarrow}{T} \cdot d\boldsymbol{s}$$

The $\boldsymbol{E}$ field on the top is given by

$$\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{r}$$

We also note that by symmetry, the force will be only along the upwards direction, z:

$$F^z = -\int T^{zi}\, ds^i$$

Thus we must calculate T^{zx}, T^{zy}, and T^{zz}:

$$T^{zx} = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\cos\phi$$

$$T^{zy} = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\phi$$

$$T^{zz} = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 (\cos^2\theta - \sin^2\theta)$$

We can then sub these into the integral, and we find that the integral gives

$$F^z = \frac{Q^2}{4\pi\epsilon_0 R^2}$$

However, we need a closed surface, we need the disc that separates the bottom and top regions:

$$F^z_{disc} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2}$$

Thus the total force will be

$$F^z = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}$$

Let us look at angular momentum. The angular momentum is defined as

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P}$$

Integrating this over all space when $\boldsymbol{P}$ is the density will give us the true total angular momentum:

$$\boldsymbol{J} = \epsilon_0\mu_0 \int \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B})\, d^3\boldsymbol{r}$$

Suppose we have a very long solenoid, with current flowing through it. The radius of the solenoid is R. The solenoid will set up a $\boldsymbol{B}$ field inside of it. We place a metal cylinder inside the solenoid, of radius a, and a second one encasing the solenoid, of radius b. We then have a potential difference across the two cylinders. This will generate an electric field. If the inner cylinder is higher potential, the $\boldsymbol{E}$ fields will point outwards from the inner cylinder to the outer cylinder. Suppose that the inner cylinder has charge Q, the outer has charge $-Q$, and the cylinders are of length l. The electric field is then given by

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0 lr}\hat{r}$$

We can also get the magnetic field:

$$\boldsymbol{B} = \mu_0 nI\hat{z}$$

We can then compute the momentum density:

$$\boldsymbol{p} = \epsilon_0(\boldsymbol{E} \times \boldsymbol{B}) = \frac{\mu_0 nIQ}{2\pi lr}\hat{\phi}$$

We can then compute the angular momentum density:

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} = -\frac{1}{2}\mu_0 nIQ(R^2 - a^2)\hat{z}$$

Suppose we gradually reduce the current. When we do this, we have an induced magnetic field (by the varying electric field), which will cause the interior cylinder to rotate, and the outer cylinder rotates in the opposite direction.

5 Electromagnetic Waves

5.1 Wave Equation

Let us look at what Maxwell's equations tell us about a vacuum, somewhere with no charge and no current. We begin with

$$\nabla \cdot \boldsymbol{E} = 0$$

$$\nabla \cdot \boldsymbol{B} = 0$$

Faraday's induction tells us that

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

And Maxwell tells us that

$$\nabla \times \boldsymbol{B} = \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t}$$

We have the trivial solution, which is that both fields are 0. Let us look for a set of solutions that are nontrivial. Starting from the third equation, let us take another cross product:

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B})$$

The fourth equation allows us to substitute into the right side:

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \boldsymbol{E}$$

We can then simplify the left side using vector identities:

$$\nabla \nabla \cdot \boldsymbol{E} - \nabla^2 \boldsymbol{E} = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \boldsymbol{E}$$

The first term is 0, and thus we have that

$$\frac{\partial^2}{\partial t^2} \boldsymbol{E} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \boldsymbol{E}$$

If we work this all out for $\boldsymbol{B}$, we get the same equation equation, just with $\boldsymbol{B}$ instead of $\boldsymbol{E}$. We have seen these equations before, these are wave equations, which we have seen before.

Matching this to the previous wave equations that we have seen, we have that the velocity of the electric and magnetic waves is

$$v = \sqrt{\frac{1}{\mu_0 \epsilon_0}} = c = 3 \times 10^8 \text{meters per second}$$

This is the speed of light.

People were not sure how the wave could travel through a vacuum, so they would hypothesize the existence of the “ether” that was the medium that the waves propagated through.

When we have a wave equation, we begin with plane waves. Plane waves are waves that are independent of two of the four coordinates (x, y, z, and t), in particular, we generally care about disregarding two of the spatial coordinates. Without loss of generality, suppose we disregard x and y, leaving just z and t. The wave equation now becomes:

$$\frac{\partial^2}{\partial t^2} \boldsymbol{E}(z, t) - c^2 \frac{\partial^2 \boldsymbol{E}}{\partial z^2} = 0$$

It turns out that the most general solution to this will be a function of the form

$$\boldsymbol{E} = \boldsymbol{E}_1(z - ct) + \boldsymbol{E}_2(z + ct)$$

Where $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are arbitrary functions of $z \pm ct$. We will talk about a special kind of plane wave, sinusoidal waves:

$$\boldsymbol{E}(z - ct) = \boldsymbol{E}_0 \sin(k(z - ct) + \delta)$$

Or the analogous cosine wave. We see that we have a kz term in the sine. If we plot the sine function at some time t, the wavelength λ, which is related to k:

$$k = \frac{2\pi}{\lambda}$$

k is essentially the number of waves within a distance of 2π, which is why it is called the wave number.

If we have kc, this is $\frac{2\pi c}{\lambda}$, this is equal to the angular frequency, ω, which is also $\frac{2\pi}{T}$. Thus we can relate the period to the wavelength, $T = \frac{\lambda}{c}$. Thus we can write the wave as

$$\boldsymbol{E} = \boldsymbol{E_0}\sin(kz - \omega t + \delta)$$

Now let us take a look at $\boldsymbol{E}_0$. Let us make $\boldsymbol{E}_0$ complex, even though we know the electric field is real. We introduce this because it simplifies calculations involving trig functions. If we do this, and drop the phase:

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(kz-\omega t)}$$

However, the physical wave will be $\text{Re}(\boldsymbol{E})$.

Thus we have a solution to the wave equation, but we still have constraints that we have to satisfy:

$$\nabla \cdot \boldsymbol{E} = 0$$

We can write this out:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

We see that for a plane wave, there is no dependence on x and y, so we can cross the first two terms out. The third term gives us that

$$ik\boldsymbol{E}_0^z e^{i(kz-\omega t)} = 0$$

Thus the only choice to make this true is to have $\boldsymbol{E}_0^z = 0$. The only condition on the constant electric field is that we cannot point in the direction that the field is moving, we can only point orthogonal to it. We have that $\boldsymbol{E}$ waves are transverse waves, they are always orthogonal to the direction of motion. If we think about classical mechanics, we have that waves on a string are also transverse. However, we also have longitudinal waves, like sound, which are generated by the compression of the air molecules. Liquids cannot support transverse waves, but solids can have both transverse and longitudinal waves.

We now have the $\boldsymbol{E}$ wave, and now we need to find $\boldsymbol{B}$. We can find this with Faraday's relation:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$ik\hat{z} \times \boldsymbol{E}_0 = i\omega \boldsymbol{B}_0$$

This tells us that $\boldsymbol{B}_0 = \frac{k}{\omega}\hat{z} \times \boldsymbol{E}_0$. We see that the $\boldsymbol{B}$ field will be related to the constant $\boldsymbol{E}$ field. Let us introduce a wave vector instead of the wave number, $\boldsymbol{k}$, which is defined as the wave number times the direction of motion, which in this case is $\boldsymbol{k} = k\hat{z}$. Thus we have that

$$\boldsymbol{B}_0 = \frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega}$$

We also note that $\boldsymbol{E} \times \boldsymbol{B}$ gives the direction of propogation.

We now want to find the magnitude of $\boldsymbol{B}$:

$$B_0 = \frac{k}{\omega}E_0 = \frac{E_0}{c}$$

We see that the magnetic field doesn't interact as strongly as the electric field.

The direction of the electric field is called the polarization direction. If $\boldsymbol{E}_0$ is along the x direction, we say that it is linearly polarized in the x direction. We also have circularly polarized waves, where the polarization direction changes. If $\boldsymbol{E}_0$ is x polarized, we have that $\boldsymbol{E}_0 = (1, 0, 0)$, and similarly for y and z polarization. For circular polarization, we have left handed and right handed polarization:

$$\boldsymbol{E}_0 = \frac{1}{\sqrt{2}}(1, i, 0) \qquad \text{Left Handed}$$

$$\boldsymbol{E}_0 = \frac{1}{\sqrt{2}}(1, -i, 0) \qquad \text{Right Handed}$$

We can have a plane wave traveling in the $\boldsymbol{k}$ direction:

$$\boldsymbol{k} = (k_x, k_y, k_z)$$

In this case, we will have that

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$$

This is the most general form of the plane wave.

The relationship between ω and k is $\omega = kc$, which is known as a dispersion relation. Note that if we use the DeBroglie relationship, we would find that $\omega = \frac{\hbar^2 k^2}{2m}$, which is a less trivial relationship between the two, even though we write the plane wave the same way.

5.2 Energy in E&M Waves

The energy density is given by

$$u_{em} = \frac{1}{2}\left(\epsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2\right)$$

Here we have to be more careful, let us write the $\boldsymbol{E}$ as $\boldsymbol{E}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$, where $\boldsymbol{E}_0$ is real. This way we don't have to mess around with complex numbers. We also have that

$$\boldsymbol{B} = \boldsymbol{B}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

Where $\boldsymbol{B}_0 = \frac{1}{c}\boldsymbol{E}_0$. Inserting this into u_{em}, we have that

$$u_{em} = \frac{1}{2}\left(\epsilon_0 \boldsymbol{E}_0^2 \sin^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) + \epsilon_0 \boldsymbol{E}_0^2 \sin^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right)$$

We see that the electric and magnetic terms have the same contribution, the electric field and magnetic field each carry half the energy of the wave. We can compute the total energy:

$$U(\boldsymbol{r}, t) = \epsilon_0 \boldsymbol{E}_0^2 \sin^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

Visible light has frequency that is on the order of magnitude 10^{14} Hertz. Gamma rays have frequencies on the order of magnitude 10^{20} Hz. Radio waves are in the kHz, MHz region. The reason that higher internet speed has higher frequency is because we can encode more information in the wave, but this comes at the cost of increased energy carried by the wave, which is harder to deal with.

Since the frequencies are so high, we take an average, so the sine term will average out to $\frac{1}{2}$, giving us that

$$U(\boldsymbol{r}, t) = \frac{1}{2}\epsilon_0 \boldsymbol{E}_0^2$$

This is the energy of the classical electromagnetic wave.

We compute the flow of energy via the Poynting vector, and we find that

$$\boldsymbol{S} = \frac{1}{\epsilon_0}\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{\epsilon_0}(\boldsymbol{E}_0\times\boldsymbol{B}_0)\sin^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)$$

We discover that this is just

$$\boldsymbol{S} = \boldsymbol{k}u_{em}c$$

This is the energy flux (also sometimes called the intensity, I), and is a really nice result, the energy flow is in the direction of the motion of the wave.

5.3 Momentum

This energy also has momentum, which is related to the Poynting vector.

Suppose we have a solar panel, which absorbs the energy. We have a constant push from the light, which must impart momentum. This is called the radiation pressure. This is the amount of momentum transferred per unit time, and we find that the pressure is

$$P = \frac{I}{c}$$

5.4 Waves in a Medium

We can write out Maxwell's equations in a medium:

$$\nabla\cdot\boldsymbol{D} = 0$$
$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{H} = -\epsilon\mu\frac{\partial\boldsymbol{E}}{\partial t}$$

These equations have too many variables, so we supply the equations of state:

$$\boldsymbol{D} = \epsilon\boldsymbol{E}$$
$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu}$$

We can plug these back into Maxwell's equations, and we will find similar equations to before:

$$\frac{\partial^2\boldsymbol{E}}{\partial t^2} - v^2\nabla^2\boldsymbol{E} = 0$$
$$\frac{\partial^2\boldsymbol{B}}{\partial t^2} - v^2\nabla^2\boldsymbol{B} = 0$$

Where $v = \frac{1}{\sqrt{\mu\epsilon}}$. We typically relate this to the speed of light:

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}$$

Where n is known as the index of refraction:

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\mu_r\epsilon_r}$$

For most materials, $\mu_r \approx 1$, so for now we say that $n \approx \sqrt{\epsilon_r}$. This is always greater than 1, (otherwise the wave would travel faster than c).

5.5 Energy of Waves in a Medium

We can define the energy density in a medium:

$$u = \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H}\right)$$

We can now use the relation between $\boldsymbol{E}$ and $\boldsymbol{D}$ and $\boldsymbol{B}$ and $\boldsymbol{H}$:

$$u_{em} = \frac{1}{2}\left(\epsilon \boldsymbol{E}^2 + \frac{1}{\mu}\boldsymbol{B}^2\right)$$

We can then define the Poynting vector:

$$\boldsymbol{S} = \frac{1}{\mu}(\boldsymbol{E} \times \boldsymbol{B})$$

For a plane wave in a medium, we have that

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

Where we use the tilde to denote a vector with complex values. Thus we have that

$$\tilde{\boldsymbol{B}} = \frac{1}{v}(\hat{k} \cdot \tilde{\boldsymbol{E}}_0) e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

We can then compute the Poynting vector:

$$I = |\boldsymbol{S}| = \frac{1}{2\mu}\frac{1}{v}E_0^2$$

If we multiply top and bottom by v, and we replace v^2 in the denominator with $\mu\epsilon$:

$$I = \frac{\epsilon v E_0^2}{2}$$

5.6 Reflection and Transmission

When we have a plane wave that is moving across a boundary between mediums, we have some segment of the wave that reflects back, and some portion that is transmitted. We can focus for now on the $\boldsymbol{E}$ field. Suppose that the $\boldsymbol{E}$ field points upwards, in the x direction, and the incoming wave is traveling to the right, in the z direction. The incident wave is thus

$$\boldsymbol{E}_{\text{Incident}} = \tilde{\boldsymbol{E}}_{I0} e^{k_1 z - \omega t}$$

Where k_1 is the wave-number in the first medium. We then have some reflected wave:

$$\boldsymbol{E}_{\text{Reflected}} = \tilde{\boldsymbol{E}}_{R0} e^{-k_1 z - \omega t}$$

We note that the $\boldsymbol{B}$ field now points into the page, rather than out of the field, since the direction has flipped.

The transmitted wave is given by

$$\boldsymbol{E}_{\text{Transmitted}} = \tilde{\boldsymbol{E}}_{T0} e^{i(k_2 z - \omega t)}$$

We see that the frequency does not change, k changes.

We now can find the boundary conditions, and use them as constraints. The first constraint that we have is $\nabla \cdot \boldsymbol{D} = 0$. We generate a matchbox, and we find that

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$$

From the fact that

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \to E_1^{||} = E_2^{||}$$

From the third condition, $\nabla \times \boldsymbol{B} = 0$:

$$B_1^{\perp} = B_2^{\perp}$$

And finally, the third condition tells us that

$$\frac{1}{\mu_1} B_1^{||} = \frac{1}{\mu_2} B_2^{||}$$

Applying the first condition, we realize that there is no $\boldsymbol{E}$ in that direction, so this tells us nothing. The second equation tells us that the parallel components have to be equal. This means that

$$E_{I0} + E_{R0} = E_{T0}$$

Once again, since the $\boldsymbol{B}$ fields have no perpendicular components, the third equation tells us nothing.

Finally, the fourth equation tells us that

$$B_{I0} - B_{R0} = \frac{\mu_1}{\mu_2} B_{T0}$$

Now we leverage the fact that the Bs are related to the Es in the plane wave. We can write this then as

$$\frac{1}{v_1}(E_{I0} - E_{R0}) = \frac{\mu_1}{\mu_2}\frac{1}{v_2} E_{T0}$$

We have two equations, and two unknowns.

If we go through and solve for the two unknown coefficients, we find that

$$E_{R0} = \left(\frac{1-\beta}{1+\beta}\right) E_{I0} \qquad E_{T0} = \left(\frac{2}{1+\beta}\right) E_{I0}$$

Where $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} \approx \frac{n_2}{n_1}$. When $\beta > 1$, this means that $n_2 > n_1$, the second medium is more dense than the first. This means that E_{R0} will have a negative sign compared to E_{I0}, a phase change of 180 degrees.

If $\beta < 1$, then the second medium is less dense ($n_2 < n_1$), we have no phase difference between the reflected wave and the incident wave.

We can calculate the ratio of the intensity of the reflected wave to the incident wave intensity, known as the reflection coefficient:

$$R = \frac{I_R}{I_I} = \left(\frac{1-\beta}{1+\beta}\right)^2 = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2$$

And similarly for the transmission coefficient:

$$T = \frac{I_T}{I_I} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

Intuitively, these must sum to 1:

$$R + T = 1$$

And indeed, these do sum to 1.

Suppose that we have a boundary, and we shoot an incident wave at some angle relative to the normal of the boundary, θ_I. We have some reflected wave, at angle θ_R, and a transmitted wave, at angle θ_T. When we do this, we have a more complex system, where the $\boldsymbol{E}$ is no longer orthogonal to the boundary. We have that

$$\boldsymbol{E}_I = \boldsymbol{E}_{I0} e^{i(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t)}$$

and similarly for $\boldsymbol{E}_R$ and $\boldsymbol{E}_T$:

$$\boldsymbol{E}_R = \boldsymbol{E}_{R0} e^{i(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega t)}$$

$$\boldsymbol{E}_T = \boldsymbol{E}_{T0} e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega t)}$$

We know that $\omega = k_I v_1 = k_R v_1 = k_T v_2$, which tells use the magnitude of the wave vectors, $k_I = k_R$, $k_T = k_I \frac{v_1}{v_2} = k_I \frac{n_2}{n_1}$.

By continuity, we must have that

$$\boldsymbol{k}_I^{\perp} \cdot \boldsymbol{r}_{\perp} = \boldsymbol{k}_R^{\perp} \cdot \boldsymbol{r}_{\perp} = \boldsymbol{k}_T^{\perp} \cdot \boldsymbol{r}_{\perp}$$

And likewise

$$\boldsymbol{k}_I^{||} \cdot \boldsymbol{r}_{||} = \boldsymbol{k}_R^{||} \cdot \boldsymbol{r}_{||} = \boldsymbol{k}_T^{||} \cdot \boldsymbol{r}_{||}$$

From these, we have that θ_I must be the same as θ_R, which is known as the law of reflection. Also from these, we can derive Snell's Law:

$$n_2 \sin\theta_T = n_1 \sin\theta_I$$

If we do everything out, we find that

$$\boldsymbol{E}_{R0} = \frac{\alpha - \beta}{\alpha + \beta} \boldsymbol{E}_{I0}$$

$$\boldsymbol{E}_{T0} = \frac{2}{\alpha + \beta} \boldsymbol{E}_{I0}$$

Where $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$.

5.7 Dispersion and Dissipation

Dispersion tells us that the index of refraction varies based on the frequency. This is what is shown in the famous Newton prism, where the different frequencies in the white light split because they each have different indices of refraction. The higher the frequency, the higher the index of refraction.

Dissipation is much easier to understand. When light strikes an object, some of it is absorbed. For example, most of the light passes through glass and water, but other things like metals, walls, floors, etc, do not allow us to see past them.

To understand these, we have to have a model for materials. Each piece of material consists of lots of atoms, which have electrons. When the electromagnetic waves strike the material, the electron is going to be influenced by the electric field. It will also be influenced by the magnetic field, but the motion of the electron is much less than c, in fact $v < \alpha c$, and thus the motion is non-relativistic, so the magnetic effects are suppressed.

When we shine the electromagnetic wave onto the electrons in the material, let us assume that the electrons can only move in 1 dimension, not 3. We know that the electron is bound to the atom, so it must feel a force from the nucleus, so we assume that it is the simplest force that we can think about that fits this, the harmonic oscillator force. We thus have that the motion of the electron in this direction will be

$$m\ddot{x} + \omega_0^2 x = 0$$

However, as the electron moves, it must radiate off energy, and thus we have an anharmonic oscillator, with a damping force:

$$m\ddot{x} + \gamma\dot{x} + m\omega_0^2 x = 0$$

Where γ takes on the role of friction in the classical oscillator case.

When the atom sees the electromagnetic wave, then we have to add on the force due to the wave:

$$m\ddot{x} + \gamma\dot{x} + m\omega_0^2 x = E_0 q e^{-i\omega t}$$

In solving this equation, we realize that we have a driving and intrinsic frequency, ω and ω_0 respectively. We assume that the solution will be

$$x = x_0 e^{-i\omega t}$$

If we insert this back into our equation:

$$(-m\omega^2 - i\gamma\omega + m\omega_0^2)x_0 = E_0 q$$

Thus we can solve for x_0:

$$x_0 = \frac{E_0 q}{-m\omega^2 - i\gamma\omega + m\omega_0}$$

$$x_0 = \frac{-E_0 q}{m}\frac{1}{\omega_0^2 - \omega^2 - i\frac{\gamma}{m}\omega}$$

Now we can compute the dipole moment of the electron, which is the charge times the distance:

$$\boldsymbol{p} = q x_0 e^{-i\omega t} = -\frac{E_0 q^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\frac{\gamma}{m}\omega}e^{-i\omega t}$$

Let us assume we have more than one electron, each having some different intrinsic frequency:

$$\boldsymbol{p}_{total} = -\frac{E_0 q^2}{m}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\frac{\gamma}{m}\omega} e^{-i\omega t}$$

Now suppose we have N electrons, we can compute the polarization density:

$$\boldsymbol{P} = N\boldsymbol{p}_{total} = -\frac{NE_0 q^2}{m}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\frac{\gamma}{m}\omega} e^{-i\omega t}$$

From this, we can compute the susceptibility, $\boldsymbol{P} = \chi_e \boldsymbol{E}$:

$$\chi_e = \frac{q^2 N}{m_e}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\frac{\gamma}{m}\omega} e^{-i\omega t}$$

Once we have this, we can compute the relative dielectric constant:

$$\epsilon_r = 1 + \chi_e = 1 + \frac{q^2 N}{m_e}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\frac{\gamma}{m}\omega} e^{-i\omega t}$$

We can relate this to the index of refraction:

$$n = \sqrt{\epsilon_r}$$

Let us now see what this model tells us. For the moment, let us assume that $\gamma \ll 1$. If so, then the dielectric is completely real. thus we have that $\epsilon_r(\omega)$. However, let us begin by assuming that ω is very small compared to ω_i and other components. Thus we can do a Taylor expansion:

$$\frac{1}{\omega_i^2 - \omega^2} = \frac{1}{\omega_i^2\left(1 - \frac{\omega^2}{\omega_i^2}\right)} \approx \frac{1}{\omega_i^2}\left(1 + \frac{\omega^2}{\omega_i^2}\right)$$

Thus we can write out the relative dielectric constant:

$$\epsilon_r = 1 + \frac{q^2 N}{m_e}\sum_i \frac{f_i}{\omega_i^2}\left(1 + \frac{\omega^2}{\omega_i^2}\right) = 1 + \frac{q^2 N}{m_e}\sum_i \frac{f_i}{\omega_i^2} + \omega^2 \frac{q^2 N}{m_e}\sum_i \frac{f_i}{\omega_i^2}$$

If we do this, and we plot n as a function of ω (by taking the square root of the expression for ϵ_r), we see that n increases as ω increases. This was discovered by Cauchy, who found that

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right)$$

Where λ is the wavelength, and A and B are coefficients. Thus we have that our model fits empirical observation.

What happens if we send in a wave that is very close to ω_i? Suppose that we have only one electron:

$$\chi_e = \frac{q^2 N}{m_e}\frac{f_i}{\omega_i^2 - \omega^2 - i\frac{\gamma\omega}{m}}$$

We can write down the real and imaginary parts, by multiplying by the conjugate of the denominator:

$$= \frac{q^2 N}{m_e} \frac{f_i}{(\omega_i^2 - \omega^2)^2 + \frac{\gamma^2\omega^2}{m^2}} \left(\omega_i^2 - \omega^2 + \frac{i\gamma\omega}{m}\right)$$

If we look at the imaginary part, and we plot $\chi_e(\omega)$, around ω_i we will have a peak at ω_i. If we look at the real part, it will be increasing, and then switch to decreasing and cross 0 at $\omega = \omega_i$, and then switch to increasing again afterwards. It turns out that the imaginary part is the absorption of the light, it peaks at the region around ω_i, which is known as resonance absorption. The real part is related to the index of refraction, and the region in which we are close to ω_i and we have a sharp decrease, is the abnormal region.

Let us assume that we have a vacuum touching a medium, which has some index of refraction that has a real part and an imaginary part:

$$n = n_R + in_I$$

EM waves in the vacuum will obey

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial z^2}$$

Which has solutions that we know:

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(kz-\omega t)}$$

If we insert this back into the equation, we will find that

$$k^2\mu\epsilon = \omega^2$$

And we have that the wave velocity is given by

$$v = \sqrt{\frac{1}{\mu\epsilon}}$$

Which gives us that

$$v = \frac{k}{\omega}$$

However, we now have a complex wave vector, since ϵ is complex:

$$\tilde{k} = k + i\kappa$$

If we go through the algebra, we can compute κ:

$$\kappa = \frac{Nq^2\omega^2}{m\epsilon_0 c} \sum_i \frac{f_i\gamma_i}{(\omega_i^2 - \omega^2)^2 + \gamma_i^2\omega^2}$$

We can now write out the electric field:

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i((k+i\kappa)z-\omega t)} = \boldsymbol{E}_0 e^{-kz} e^{i(kz-\omega t)}$$

We see that we have exponential decay when we enter the medium. This is absorption, the field decays as it travels through the medium. We can define the dissipation length:

$$\alpha = \frac{1}{2\kappa}$$

This quantifies how long it takes for the electromagnetic wave to disappear in the medium. We see that this is dependent on the frequency, since $\kappa(\omega)$. This is why certain frequencies can penetrate further into the same material than other frequencies.

Let us look at water, and try to find the dissipation length as a function of frequency. Since the molecular structure of water is so complicated, the function for κ is extremely complicated, but roughly speaking, we see a lot of variation based on ω. We have two peaks, and right in between these two peaks, we have the window for visible light, where water does not absorb the electromagnetic waves.

Let us do a little summary. We are solving the equation

$$\nabla^2 \boldsymbol{E} - \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{dt^2} = 0$$

This is the wave equation for the electric field in a medium. In a medium, we have that $\epsilon = \epsilon_0 \epsilon_r = \epsilon_0 n^2$. Thus we have that $\mu\epsilon = \frac{n^2}{c^2} = \frac{1}{v^2}$, where v is the speed of the wave through the medium. When we talk about dispersion, this is when ϵ_r is a function of ω, the speed of propogation depends on the frequency. This also affects the refraction angle. However, not only do we have a real part, we also have an imaginary part:

$$\epsilon_r(\omega) = \epsilon_r^R(\omega) + i\epsilon_r^I(\omega)$$

If we allow the electric field to have real and imaginary components, and we insert this definition of ϵ_r into the wave equation, we will find that $k^2 - \mu\epsilon\omega^2 = 0$, which tells us that $\tilde{k}^2 = \mu\epsilon\omega^2$, where we have used the tilde to denote the fact that it is complex. Thus we have that

$$\tilde{k} = k + i\kappa$$

We can solve for what κ is:

$$\kappa = \frac{\mu_0 \epsilon_r^I \omega^2}{2k}$$

This κ is important because it affects the wave:

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-\kappa z} e^{i(kz - \omega t)}$$

We see that the κ determines the strength of the dampening force. The strength of the wave depends on the depth into the surface of a material. We define a penetration depth:

$$\lambda(\omega) \propto \frac{1}{\kappa}$$

κ is also known as the absorption coefficient.

5.7.1 Metals

Let us look at metal mediums. When we are inside a metal, we cannot have any free charges, they are all driven to the surface. However, we can have a current inside a metal:

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

Where σ is the conductivity. Perfect metals would have $\sigma = \infty$.

If we now look at Maxwell's equations in metal, and go through all the algebra, we arrive at the metallic wave equation:

$$\nabla^2 \boldsymbol{E} = \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{dt^2} + \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t}$$

If we do out all the math related to k once more, we find that

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

This is the dispersion relation for EM waves in metallic mediums. This dispersion relation is very similar to the one we previously saw, we still have an imaginary part. We can massage it to look similar to what we had before:

$$\tilde{k}^2 = \mu\epsilon_{\text{eff}}\omega^2$$

Where $\epsilon_{\text{eff}} = \epsilon + \frac{i\sigma}{\omega}$. If we think back to the spring model for a medium, we can think of metals as being made up of springs with no restoring force, which makes sense, since the electrons are free. We can go through all the algebra and compute κ:

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}} \left(\left(1 + \frac{\sigma^3}{\omega^2\epsilon^2}\right)^{1/2} - 1 \right)$$

In a metal, the skin depth is similar to the penetration depth:

$$d \propto \frac{1}{\kappa}$$

This is generally very small.

If we think about how power is transported through wires, when power propagates along power lines, it is transported through a current along the surface.

5.7.2 Reflection and Penetration on Metals

Suppose we have an electromagnetic wave striking a metal. Some amount of it will penetrate into the metal, and some of it will be reflected back. We can compute the reflection and transmission coefficients, by looking at boundary conditions and Maxwell's Equations. It turns out that if we have a perfect conductor, the waves are completely reflected back, the better the conductor, the more light is reflected back. This is the principle behind mirrors. For this reason, we often use silver for mirrors, because it has a high conductivity.

5.8 Gauge Symmetry

We have Maxwell's equations in terms of $\boldsymbol{E}$s and $\boldsymbol{B}$. We have studied this in an area where there is no σ, and there is no $\boldsymbol{J}$. We have studied them without thinking about where they came from. When we add the current and charge densities into Maxwell's equations, it turns out that its much better to work with potentials:

$$\nabla \cdot \boldsymbol{E} = -\frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = -\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{J}$$

If we now use the fact that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, we find that the third equation now becomes

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0$$

This tells us that we can write these as the gradient of some scalar field:

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla\phi$$

Thus we can write out the electric and magnetic fields in terms of the vector potential and the new scalar potential:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

Thus we have a potential in 4 dimensions, $\boldsymbol{A}$ covers 3 dimensions of space, and the scalar field adds another. This is naturally consistent with Einstein's theory of special relativity. In fact, all of electromagnetism is consistent with special relativity, as we have 4 equations that can be written in terms of special relativity. If we introduce a 4D vector, $x^\mu = (\boldsymbol{x}, t)$, and let $(\boldsymbol{A}, \phi) = A^\mu$, we can write that

$$F_{\mu\nu} = d_\mu A_\nu - d_\nu A_\mu$$

And

$$d_\mu F^{\mu\nu} = J^\nu$$

All of Maxwell's theory implies special relativity.

$\boldsymbol{E}$ and $\boldsymbol{B}$ are physical, they are measurable. $\boldsymbol{A}$ and ϕ on the other hand are fully mathematical, they are not physically. A question that we can ask is, if we have a given set of $\boldsymbol{E}$ and $\boldsymbol{B}$, does that uniquely determine $\boldsymbol{A}$ and ϕ? It turns out that we cannot uniquely determine them, we can create a new one of each just by adding a gradient:

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla\psi = \boldsymbol{A}'$$

$$\phi \to \phi - \frac{\partial\psi}{\partial t} = \phi'$$

We can insert these into Maxwell's equations:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}'}{\partial t} - \nabla\phi' = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\frac{\partial\psi}{\partial t} - \nabla\phi + \frac{\partial}{\partial t}\nabla\phi$$

Where we have cancellation:

$$= -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi$$

As we expected from Maxwell's equations.

This invariance is known as gauge symmetry, and we see gauge symmetries quite often in physics. It turns out that if we write down a theory with a vector field like A^μ, and you constrain the system to have a certain gauge symmetry, we can extract Maxwell's equations from the theory. This particular case is called a $U(1)$ gauge theory.

If instead of a single A^μ, we had 8, we see that we have QCD, which governs the strong interaction. If we had 4, we'd have the weak interaction.

We often impose a "fixed gauge" condition, where we try to pick the best choices for $\boldsymbol{A}$ and ϕ to make our lives easier. One common choice of imposed condition is that $\nabla \cdot \boldsymbol{A} = 0$. This is known as the Coulomb Gauge, which is a weird name, since Coulomb had no idea about any of this stuff.

Another gauge we can pick is that

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0$$

This is known as the Lorentz Gauge, or the Lorenz Gauge, which are named after two different people (Our textbook uses Lorenz).

There also exists the axial gauge, where the z component of A is 0:

$$A^z = 0$$

And the temporal gauge, where

$$\phi = 0$$

If we use the Lorenz gauge, and insert the definition into Maxwell's equations, we can find that

$$\left[\nabla^2 - \frac{1}{\epsilon_0 \mu_0}\frac{\partial^2}{\partial t^2}\right]\boldsymbol{A} = \mu_0 \boldsymbol{J}$$

If we do this with ϕ:

$$-\left[\nabla^2 - \frac{1}{\epsilon_0 \mu_0}\frac{\partial^2}{\partial t^2}\right]\phi = \frac{\rho}{\epsilon_0}$$

We see that we have two decoupled differential equations. Looking at the equation for ϕ, we know that if everything is time independent, we will end up with just Coulomb's Law:

$$\phi(r) = \iiint \frac{\rho(r')}{4\pi\epsilon_0 |r - r'|}\, d^3 r'$$

Now let us make a naive guess, and add time dependence on both sides:

$$\phi(r, t) = \iiint \frac{\rho(r', t)}{4\pi\epsilon_0 |r - r'|}\, d^3 r'$$

This can be immediately dismissed, because this implies that if something changes in a charge distribution, it instantly affects distant points, which we know cannot be true. We need some propagation speed for the effect of the change in charge.

Let us suppose we have a point charge, located at r_0, which is a function of t:

$$r_0(t) = v_0 t$$

Suppose we are interested about the field at a point (r, t). We claim that there is some point in time t_r, such that the time it takes for the particle to get from the previous point to the current point is the same as the time it takes for the light to travel to the point r:

$$(t - t_r) = \frac{|\boldsymbol{r} - \boldsymbol{r}_0(t_r)|}{c}$$

If we use this instead of t in our naive guess, we find that we actually get what is known as the retarded potential:

$$\phi(r,t) = \iiint \frac{\rho(r', t_r)}{4\pi\epsilon_0 |r - r'|}\, d^3r'$$

However, we also have some advanced position in the future, t_a. By the time the particle moves from its current position to the advanced position, the light from the point r will have reached the point t_a. This gets us an advanced potential. However, if we take this into account, we break causality, so we discard this solution. Let us talk about some specific cases related to a moving point charge.

For a fixed charge, we can express the potential as

$$\rho = q\delta(\boldsymbol{r} - \boldsymbol{r}_0)$$

The naive idea would be to just add time dependence (specifically dependence on t_r) to $\boldsymbol{r}_0$, but that does not work, because the claim

$$\iiint \rho\, d^3r' = Q$$

no longer holds true. The mathematical reason depends on the fact that the delta function is normalized in a specific way. Physically, suppose we have a rectangle, moving with velocity v. If we look at the time it takes for the light emitted by the front, and the light emitted by the back, it turns out that due to the propagation speed of the light, the length of the rectangle that we observe will not be the actual length, it will be larger by a factor ΔL:

$$L_{obs} = L + \Delta L = \frac{c}{c - v} L$$

If we generalize this to any angle of observation:

$$L_{obs} = \frac{c}{c - \boldsymbol{v} \cdot \hat{n}} L$$

This tells us that the observed charge will be larger than the actual size. The real potential is given by

$$\rho = q \frac{c}{c - \boldsymbol{v}(t_r) \cdot \boldsymbol{n}} \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_0(t_r))$$

We can also use the substitution $\frac{v}{c} = \beta$. If we use this ρ to solve for ϕ:

$$\phi(r,t) = \frac{q}{4\pi\epsilon_0 \left(|\boldsymbol{R}| + \boldsymbol{\beta} \cdot \boldsymbol{R}\right)}$$

where $\boldsymbol{R}$ is a function of t_r, as is $\boldsymbol{\beta}$. This is known as the Lienard-Wiechert potential for a moving point charge.

When we think of the $\boldsymbol{E}$ field, we have a bunch of derivatives:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

These derivatives are derivatives with respect to t, but we have everything in terms of t_r, which is related by the equation

$$t_r = t - \frac{|\boldsymbol{r} - \boldsymbol{r}_0(t_r)|}{c}$$

Thus we have to do everything carefully:

$$-\nabla\phi = \frac{1}{4\pi\epsilon_0}\frac{1}{(|\boldsymbol{R}| - \boldsymbol{\beta}\cdot\boldsymbol{R})^2)}\cdot\nabla(|\boldsymbol{R}| - \boldsymbol{\beta}\cdot\boldsymbol{R})$$

If we solve for $\boldsymbol{A}$, similar to the way we did for ϕ, we will find that

$$\boldsymbol{A} = \frac{1}{4\pi(\boldsymbol{R} - \boldsymbol{\beta}\cdot\boldsymbol{R})}\times\mu_0 q\boldsymbol{v}(t_r)$$

If we go through a bunch of work, we eventually would reach the result that

$$\boldsymbol{E}(r,t) = \frac{1}{4\pi\epsilon_0}\frac{1}{(|\boldsymbol{R}| - \boldsymbol{R}\cdot\boldsymbol{\beta})^3}\left[(1-\beta^2)(\boldsymbol{R} - |\boldsymbol{R}|\boldsymbol{\beta}) + \boldsymbol{R}\times(\boldsymbol{R} - \boldsymbol{\beta}(|\boldsymbol{R}|))\times\frac{a}{c^2}\right]$$

If we have a charge moving at some speed $\ll c$, very far away, we will have that $\beta \ll 1$, and $|R| \to \infty$. We will find that

$$\boldsymbol{E}(r,t) = \frac{q}{4\pi\epsilon_0}\frac{1}{|R|^3}\left[\boldsymbol{R} + \boldsymbol{r}\times\left(\boldsymbol{R}\times\frac{a}{c^2}\right)\right]$$

The first term here goes as $\frac{1}{|R|^2}$, which is why we call it the Coulomb term. The second term goes as $\frac{1}{|R|}$, and is known as the radiation term. We note that this term depends on a, the acceleration of the charge. If the charge is not moving, it does not radiate. This leads to the famous argument, that accelerating charges radiate. This led to a crisis in early quantum mechanics, where if we have a hydrogen atom, with an electron orbiting around it, the power will be given by

$$\boldsymbol{P} = \int \boldsymbol{S}\cdot d\boldsymbol{a} \propto \int \boldsymbol{E}\times\boldsymbol{B} \propto e^2 a^2$$

In fact, we will find that $P = \frac{\mu_0}{6\pi c}e^2a^2$. This is known as Larmouv's formula. It turns out that classic electrodynamics predicts that the hydrogen atom should have a lifetime of 10^{-11} seconds, which we know is obviously not true. Quantum mechanics was built in part to explain why the electron in the hydrogen atom doesn't fall into the center of the atom.

If we look at the case when $R \to \infty$ and when $\beta \ll 1$, we see that the Coulomb term of the electric field is

$$\boldsymbol{E}_c = \frac{a}{4\pi\epsilon_0}\frac{1}{(|\boldsymbol{R}| - \boldsymbol{\beta}\cdot\boldsymbol{R})^3}(1-\beta^2)(\boldsymbol{R} - \boldsymbol{\beta}\cdot|\boldsymbol{R}|)$$

If we look at the radiation term, ignoring the prefactors and constants, we see that

$$\boldsymbol{E}_r \sim \frac{|\boldsymbol{R}|^2}{c^2}\left[\hat{R}(\hat{R}\cdot\hat{a}) - \boldsymbol{a}\right]$$

This radiation term is in the direction of $a_\perp$, when we have something radiating, it radiates perpendicular to the direction it is accelerating. This is why antennae point upwards, the signal generated upwards is the weakest, the transverse radiation is stronger.

Suppose we have a charge moving with constant velocity, $r = v_0 t$. We will find that the $\boldsymbol{E}$ field in this case will be:

$$\boldsymbol{E}(r,t) = \frac{a}{4\pi\epsilon_0}\frac{\hat{R}(t)}{|\boldsymbol{R}(t)|^2}\frac{1-\beta^2}{(1-\beta^2\sin\theta)^{3/2}}$$

Suppose that θ, the angle from the observer and the particle is 0, we see that

$$\boldsymbol{E}_{||} = \frac{a}{4\pi\epsilon_0}\frac{\hat{R}}{|\boldsymbol{R}(t)|^2}(1-^2)$$

We note that $1-\beta^2 < 1$, which gives us an answer that is smaller than what Coulomb's Law would give. If we are a perpendicular observer, $\sin\theta = 1$, and thus we have that

$$\boldsymbol{E}_{\perp} = \frac{a}{4\pi\epsilon_0}\frac{\hat{R}}{|\boldsymbol{R}(t)|^2}\frac{1}{\sqrt{1-\beta^2}}$$

This last factor is known as $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, and we note that it is larger than 1. We see that this is consistent with Lorentz symmetry from special relativity.

Suppose we have a infinitely wire of charge, moving with a velocity v, and with line charge λ. We say that by symmetry, the $\boldsymbol{E}$ field must be point to the right, since the system has translation symmetry, and $\boldsymbol{E}$ only cares about the charge, not the motion.

We can write out the integral for the electric field:

$$\int \frac{\lambda dz}{4\pi\epsilon_0}\frac{(1-\beta^2)(\boldsymbol{R}-\boldsymbol{\beta}\cdot\boldsymbol{R})}{(|\boldsymbol{R}|-\boldsymbol{R}\cdot\boldsymbol{\epsilon})^3} = \int \frac{\lambda\, dz}{4\pi\epsilon_0}\frac{\hat{R}}{|\boldsymbol{R}|^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^2}$$

We can now use the fact that $R = \sqrt{d^2+z^2}$:

$$= \int \frac{\lambda\, dz}{4\pi\epsilon_0}\frac{d(1-\beta^2)\hat{s}}{(d^2+z^2-\beta d^2)^{3/2}} = \frac{\lambda d(1-\beta^2)\hat{s}}{4\pi\epsilon_0}\frac{2}{(1-\beta^2)d^2} = \frac{\lambda\hat{s}}{2\pi\epsilon_0}d$$

Now using the fact that $\boldsymbol{B} \sim \frac{\boldsymbol{v}}{c^2}\times\boldsymbol{E}$:

$$\boldsymbol{B} = \frac{\mu_0\lambda v}{2\pi d}$$

We actually expected this result, the moving charge is the same as a steady current in the wire, which we can solve easily using Ampere's Law. If we solve via Ampere's Law, we find that the magnetic field is the exact same thing that we found.

Suppose we have a circular loop of charge, with current I flowing through it. The loop has radius R. By the Biot-Savart Law, we know that the magnetic field will be

$$\boldsymbol{B} = \frac{\mu_0\lambda\omega R}{2R} = \frac{\mu_0\lambda\omega}{2}$$

This is a harder problem than the last one, we have the centripetal acceleration acting on every bit of charge. However, we don't need to worry about t_r, since every point is equally far away from the point we are interested in.

We know that the acceleration will be:

$$\boldsymbol{a} = \frac{v^2}{|\boldsymbol{R}|}\hat{R}$$

And we know that $\boldsymbol{\beta} = \omega|\boldsymbol{r}|$. We can then write out the electric field:

$$\boldsymbol{E} = \frac{\lambda}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{R}^3|}\left[(1-\beta^2)(\boldsymbol{R}\cdot\boldsymbol{\beta}|\boldsymbol{R}|) + \boldsymbol{R}\times\left[-\boldsymbol{\beta}|\boldsymbol{R}|\times\frac{\boldsymbol{a}}{c^2}\right]\right]$$

We can then do out the cross product, and we find that it becomes $\frac{\beta}{c^2}$. If we go through all the remaining vector work, we find that

$$\boldsymbol{E} = \frac{\lambda}{4\pi\epsilon_0}\frac{\hat{R}\times\boldsymbol{\beta}}{|\boldsymbol{R}|^2}$$

5.9 Radiation

In the Lorenz gauge, we have that

$$\frac{\partial\phi}{\partial t} + \nabla\cdot\boldsymbol{A} = 0$$

If we use the 4-vector notation, where we have that $A^\mu = (\phi, \boldsymbol{A})$, we write this gauge as

$$\frac{\partial A^\mu}{\partial x_\mu} = 0$$

Once we have Maxwell's equations, we can look at their propagation through free space. this is radiation. This will give us the wave equations

$$\left(\frac{\partial^2}{\partial\boldsymbol{x}^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{bmatrix}\boldsymbol{A}\\ \phi\end{bmatrix} = 0$$

But what are the sources of these EM waves? The sources must be currents and charges, represented by $\boldsymbol{J}$s and ρs. We know that these generate electric and magnetic waves, and we can solve for the potentials easily if we don't have time dependence. The reason that time dependence does not work out as naively thought is that the waves must have time to propagate, we cannot have information that propagates faster than the speed of light. From this we found the definition of the retarded time, t_r, which accounts for the offset due to the propagation time:

$$t_r = t - \frac{R}{c}$$

From this, we can implement time dependence, and obtain the formulas for what are known as the retarded potentials in free space.

Let us now work through an example to see how waves are produced. Suppose we have some charges and current confined in a region, and they varying as a function of time. This region emits some electromagnetic waves, which means that there is a radiation power, P, which is related to the energy density, $P = \boldsymbol{E}\times\boldsymbol{B}$. If we are some distance r away from the region, the radiation power will be proportional to the Poynting vector and the surface area:

$$P \sim \int(\boldsymbol{E}\times\boldsymbol{B})\cdot d\boldsymbol{s}$$

We know that this must be a finite quantity. We also know that the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields will decay as $\frac{1}{r}$, which is a pretty slow decay. We see that the electromagnetic waves can send information pretty far.

Suppose we have a static charge Q, and we have a point r away. We know that the electric field due to the point charge decays as $\frac{1}{r^2}$, as given by Coulomb's Law. Similarly, if we have a static current, the Biot-Savart Law tells us that the magnetic field will also decay as $\frac{1}{r^2}$. Thus, it turns out that static currents and static charges can never radiate. In fact, the introduction of t_r is what makes it possible for the fields to decay as $\frac{1}{r}$.

5.9.1 Dipole Radiation

Let us make a model for a dipole. Suppose we have a positive charge $+q(t)$ on the z axis, as well as a negative charge $-q(t)$ underneath it, a distance d away. Both of these charges will depend on time:

$$q(t) = q_0 \cos(\omega t)$$

We can define a dipole moment, which is the charge times the distance between them:

$$\boldsymbol{P}(t) = dq(t) = dq_0 \cos(\omega t)\hat{z}$$

Let us compute the potential at some faraway point $\boldsymbol{r}$. We want to compute the potential $V(r,t)$ at this point. This can be done easily without t_r :

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_0\cos(\omega t)}{R_+} - \frac{q_0\cos(\omega t)}{R_-}\right]$$

But we can take into account t_r:

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_0\cos\left(\omega\left(t-\frac{R_+}{c}\right)\right)}{R_+} - \frac{q_0\cos\left(\omega\left(t-\frac{R_-}{c}\right)\right)}{R_-}\right]$$

Where R_+ is the distance to the positive charge and R_- is the distance to the negative charge:

$$R_\pm = \sqrt{r^2 \mp rd\cos\theta + \frac{d^2}{4}}$$

We can now assume that $r \gg d$, and we can Taylor expand the distance:

$$R_\pm = \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) + \dots$$

We can also say that $r \gg \lambda \sim \frac{c}{\omega}$.

If we write this out, we will find that

$$V(r) = \frac{q_0 d\cos\theta}{4\pi\epsilon_0 r}\left[-\frac{\omega}{c}\sin\left(\omega\left(t-\frac{r}{c}\right)\right) + \frac{1}{r}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\right]$$

We see that we have a term that goes as $\frac{1}{r}$, and we have a term that goes $\frac{1}{r^2}$. We can discard the terms that cannot go very far, so we just focus on the $\frac{1}{r}$ effects.

Let us now compute the $\boldsymbol{E}$ field:

$$\begin{aligned}\boldsymbol{E} &= -\nabla\phi \\ &= \frac{q_0 d\omega\cos\theta}{4\pi\epsilon_0 rc}\nabla\left[\sin\left(w\left(t-\frac{r}{c}\right)\right)\right] \\ &= \frac{P_0\omega^2}{4\pi\epsilon_0 c^2}\frac{\cos\theta}{r}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\hat{r}\end{aligned}$$

Where we have thrown away any terms that are generated that go as $\frac{1}{r^2}$, and we see that we have a term that goes as $\frac{1}{r}$, something that we could never find for stationary charges or currents, since we don't have the retarded time.

Now let us look back at our model. We have a changing charge, and in fact we generate a current oscillating up and down. From this, we have an $\boldsymbol{A}$ field:

$$\boldsymbol{A} = -\frac{\mu_0 q_0 \omega}{4\pi r} \sin\left(\omega\left(t - \frac{r}{z}\right)\right)\hat{z}$$

This actually affects the $\boldsymbol{E}$ field:

$$\begin{aligned}\boldsymbol{E} &= -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \\ &= -\frac{\mu_0 P_0 \omega^2}{4\pi}\frac{\sin\theta}{r}\cos\left(\omega(\left(t - \frac{r}{c}\right))\right)\hat{\theta}\end{aligned}$$

And we can find the generated $\boldsymbol{B}$ field:

$$\begin{aligned}\boldsymbol{B} &= \nabla \times \boldsymbol{A} \\ &= -\frac{\mu_0 P_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\left(\omega\left(t - \frac{r}{c}\right)\right)\hat{\phi}\end{aligned}$$

If we look at the direction of the wave, $\boldsymbol{E}\times\boldsymbol{B}$, we see that this is along $\hat{r}$. We are sending the wave outwards from the dipole. We can also compute the Poynting vector:

$$\begin{aligned}\boldsymbol{S} &= \frac{1}{\mu}(\boldsymbol{E}\times\boldsymbol{B}) \\ &= \frac{\mu_0 P_0^2 \omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^3}\hat{r}\end{aligned}$$

We note that the radiation is mostly going horizontally, due to the $\sin^2\theta$, which is close to 0 when we are going vertically. The main culprit for wave production is the fact that the field cannot instantly catch up with the change in the charge density, we have the retardation time offset.

We can also compute the total dipole radiation power:

$$\begin{aligned}P_{\text{radiation}} &= \int \boldsymbol{S}\cdot d\boldsymbol{a} \\ &= \frac{\mu_0 P_0^2 \omega^4}{12\pi c}\end{aligned}$$

Note that we can also do something similar for another set up, with 4 charges, where we would have electric quadropole radiation, and likewise for magnetic quadropole radiation. However, these radiations get weaker and weaker in strength, and are controlled by a parameter, $\frac{d}{\lambda}$, which is the dimension of the system over the wavelength.

Most of the time when we have radiation, it is electric dipole radiation. This is the most efficient method of radiation, the higher order multipole radiation types are less talked about since they are less efficient. The principle behind all the multipole radiations is based on accelerating charges. When a charge moves, it only radiates when it accelerates. Classically, the radiation power is proportional to the acceleration:

$$P \propto a^2$$

At places like Brookhaven, they create a beam by keeping particles in orbits, such as synchrotons, and then shooting them out as beams. An even better light source is free electron LASERs, which is capable of generating the most synchronized/high luminosity man-made radiation.

Suppose we have two charges, q_1 and q_2. At the start of the course, we would talk about the electric fields generated, based on the force that affects the two charges. What happens if we (the observer) are moving at velocity v perpendicular to the axis between the two charges? Intuitively, we should expect nothing to change. However, when we move, we should expect a current, which should induce a magnetic field. In fact, the faster that we move, the stronger the magnetic field induced. When we move, we see a magnetic field that the static observer does not see, as well as a different electric field, not the static field. We expect that the force should be the same, no matter the frame that we are in.

Suppose Alice is static, with some charge, and Bob is moving in a rocket with another charge. Can we establish a relationship between what Alice sees, $\boldsymbol{E}$ and $\boldsymbol{B}$, and what Bob sees, $\boldsymbol{E}'$ and $\boldsymbol{B}'$? It turns out that we can, and the transformation in fact tells us that space and time are not independent, like in Newtonian mechanics.

Einstein's relativity states that space and time is a 4D space:

$$x^\mu = (ct, \boldsymbol{x}) \;\; \mu = 0, 1, 2, 3$$

And we can transform between frames via a matrix operation:

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

Which is a rotation in this 4D space (Λ is a Lorentz transformation matrix). In Euclidean space, the length of a vector is given by

$$\boldsymbol{x}^2 = x_1^2 + x_2^2 + x_3^2$$

But this is not the case for 4D vectors:

$$x^\mu x_\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

Where the signs are based on the convention (Above is the Stanford/West Coast convention). This space is known as a Minkowski space.

We can define the potential:

$$A^\mu = (c\phi, \boldsymbol{A})$$

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial^\nu A^\mu}{\partial x_\nu}$$

This is anti-symmetric, so $F^{\mu\nu} = -F^{\nu\mu}$. Because of this, $\mu \neq \nu$. From the original 16 components, this anti-symmetry cuts us down to just 6 components. 3 of those come from the $\boldsymbol{E}$ field, and 3 come from the $\boldsymbol{B}$ field. If we look at a specific component, F^{01}, and look at the formula:

$$F^{01} = -\frac{\partial A^1}{\partial t} - \frac{\partial A^0}{\partial x^1}\boldsymbol{E}$$

Which is just what Maxwell's equations tell us about the electric field.

Going back to the Alice and Bob case, we cna transform between two frame's force tensors:

$$F_B^{\mu\nu} = \Lambda_\alpha^\mu \Lambda_\beta^\nu F_A^{\alpha\beta}$$

Where again the Λ matrices are the Lorentz transformation matrices.

This means that electromagnetism inherently obeys special relativity. Special relativity is a consequence of switching frames in electromagnetism. If we know the physics in one frame, we can use special relativity to find the physics in any other frame. Lorentz discovered the transformation before special relativity was invented, Lorentz was just trying to figure out how to keep electromagnetism constant between frames.

If we expand current to 4 dimensions:

$$J^\mu = (\rho, \boldsymbol{J})$$

and we can write out the charge conservation law as

$$\partial_\mu J^\mu = 0$$

And we can write one equation that governs all of electromagnetism:

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu}$$

Herman Weyl said that the physics should not depend on the frame, but also should not depend on rotation. This led to the start of the understanding of fundamental interactions using gauge theories. He stated that charges should be invariant under $U(1)$ gauge transformations, and all the physics of electromagnetism falls out from there.