PHYS851 Notes

Hersh Kumar

Contents

1 Review of QFT I

We have 2 problems with quantum mechanics. The first is Lorentz invariance, quantum mechanics is not relativistic, and it is sort of "hard coded" into QM that we do not obey relativity. This is because relativity treats space and time on equal footing, we can rotate x into t via a boost. However, quantum mechanics treats x and t very differently, x is an operator, we can compute the expected location of things by taking an expectation value of position, $\langle x \rangle$. On the other hand, t is a label, not an operator. This difference between space and time is baked into quantum mechanics. The second problem is that we have superluminal travel, which according to relativity, removes causality from the theory, which is a massive problem. Explicitly:

$$\langle x_f | e^{-iHt} | x_i \rangle \neq 0 \quad \forall x_f,\, t > 0$$

These are our two reasons for requiring drastic changes to the quantum mechanics paradigm.

The approach we take to remedying these problems is to look at a classical theory in which space and time are taken as labels, classical electromagnetism. If we do this, we accidentally end up solving the second problem! We can first pick what we are describing as a function of space and time, and then we can quantize the theory to make it quantum mechanical. In classical mechanics, we have many examples of things that can be functions of x and t, such as potentials, electric fields, the stress-energy tensor, etc. Thus we have scalars, vectors, tensors, that are all functions of space and time. In field theory, we systematize this construction. Why do we like vectors? Because they transform nicely under rotations. In the relativistic case, we need things that transform nicely under rotations *and boosts*. What do we mean by "nice" here? Looking at this mathematically, rotations form the group $SO(3)$, and the addition of boosts constructs the group $SO(3,1)$[1]. This group has some special properties, the first of which is that we can write it as two rotation groups:

$$SO(3,1) = SO(3) \otimes SO(3)$$

That is, we can write our Lorentz group elements in terms of objects from both rotation groups:

$$R_{x,y,z} \pm iK_{x,y,z} \in SO(3,1)$$

Let us now consider a single rotation group. What does it mean to be nice under $SO(3)$ (sometimes written as $SU(2)$). What we mean intuitively is that we multiply a vector position by a rotation matrix:

$$\boldsymbol{x}' = R\boldsymbol{x}$$

or, in index notation:

$$x'_a = R_{ab} x_b$$

This is fundamentally what we mean when we say that things behave "nicely" under rotations. However, the key point is that the matrices R_{ab} are very specific matrices, they must be representations of the rotation group.

Simple examples can show that the order of rotations matter, the rotation matrices do not commute:

$$R_x(\theta_1) R_y(\theta_2) \neq R_y(\theta_2) R_x(\theta_1)$$

[1] Note that we do not write $SO(4)$, because we have the restriction on $\boldsymbol{x}^2 - t^2$, the minus sign here matters

Any rotation matrices that we define must satisfy the infinite number of commutation constraints that we expect from 3 dimensional intuition. We can simplify this down to a finite number of constraints. First, we note that we can define any rotation as:

$$R = e^{i\theta_j \tau_j}$$

Where τ_j is a Hermitian matrix, since rotations must be length preserving, and $j \in x, y, z$. With this representation of rotations, the constraints on rotations boil down to the statement:

$$[\tau_j, \tau_k] = i\varepsilon_{jkl}\tau_l$$

In other words, if we want an $N \times N$ representation of the rotations, we need 3 $N \times N$ matrices that satisfy this condition.

Note that for non-rotation groups, the only difference is that instead of the Levi-Civita symbol, we have some other number, generally know as the structure constant:

$$[\tau_j, \tau_k] = if_{jkl}\tau_l$$

Returning to rotations in particular, it turns out that there is 1 solution, for all N. What does this mean for us? For $N = 1$, $\tau_i = 0$, and we have a solution. For $N = 3$, we recover vectors, and for $N = 5$ we recover anti-symmetric matrices. Classically, we recover the odd N solutions.

What about even N? For $N = 2$, we find that $\tau = \sigma$, the Pauli matrices. Why did we miss this in classical mechanics? It's because rotations of 360 degrees using the Paulis produces a minus sign, rather then the identity. In the quantum mechanical case, the minus sign doesn't matter, since we only care about the modulus of the wavefunction.

Now let us look back at the Lorentz group, which we said was composed of two rotation groups, generally labelled by left and right rotation groups:

$$SO(3,1) = SU(2)_L \otimes SU(2)_R$$

For every object we construct, we have to know how it transforms under each rotation group, which we generally label by the maximum t_z eigenvalue for each of the groups. For example, we could have an object be a scalar under each group, which we label as $(0, 0)$. We could have it be a scalar under one group, but a fermion under the other:

$$\text{Left fermions} \to \left(\frac{1}{2}, 0\right)$$
$$\text{Right fermions} \to \left(0, \frac{1}{2}\right)$$

We could also have a fermion under both groups, which turns out to be a vector:

$$\text{Vector} \to \left(\frac{1}{2}, \frac{1}{2}\right)$$

We can keep going for all cases, and its up to experiment to tell us which ones we have discovered in reality.

We have now categorized every object that we can write down that correctly transforms under the symmetry group that we care about. Note that nothing we have done so far is quantum, so let

us now introduce it. First, we must determine which "language" to use. Classically, we have two formalisms, Lagrangian/Hamiltonian mechanics and via equations of motion, such as Newton's Laws or Maxwell's equations.

It turns out that the Lagrangian/Hamiltonian formalism is much easier to work with. Consider computing something of the form

$$\langle x_f | e^{-iHt} | x_i \rangle$$

The way to compute this is to sum up the possible paths by which our particle can propagate:

$$\begin{aligned}\langle x_f | e^{-iHt} | x_i \rangle &= \sum_{\text{all paths}} e^{i\phi} \\ &= \int \mathrm{d}x_{\text{all paths}} e^{iS}\end{aligned}$$

Where S is the action. When we are doing Lagrangian mechanics classically, we are really just doing this and taking the most probable path as the classical trajectory (principle of least action). Quantum mechanically, we have to take into account the other paths. For this reason, Lagrangians and Hamiltonians are more natural. There are also some other benefits, such as the fact that if $\mathcal{L}$ has a symmetry, the theory described by this $\mathcal{L}$ also has that symmetry.

To quantize a theory, recall that we impose the canonical commutation relations:

$$[q, p] = i\hbar$$

Note that this is natural in the case of the Lagrangian formalism, since $\mathcal{L}$ is a function of q and $\dot{q}$. In the case of electromagnetism, $q = A_\mu$, in the case of scalar field theories, $q = \phi$, in the case of gravity $q = h_{\mu\nu}$, etc.

Recall that the classical EM Lagrangian is given by

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

Where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. To quantize this theory, we first find that $p_\mu = \dot{A}_\mu$, and therefore we impose the commutation relations[2]:

$$\left[A_\mu(x), \dot{A}_\nu(y)\right] = -i\delta^3(x-y)\, g_{\mu\nu}$$

If we do this, we find that it is most natural to define creation and annhilation operators:

$$a, a^\dagger \sim x \pm ip$$

It turns out that we can express these in terms of Fourier transforms:

$$\phi = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a^\dagger_{\boldsymbol{p}} e^{ip\cdot x} \right)$$

Where the creation and annhilation operators obey the commutation relations:

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right] = (2\pi)^3 \delta^3(p-q)$$

[2]Note that we are using the $(+,-,-,-)$ metric convention.

Note that in the case of a vector particle, we would write the same Fourier expansion, just with additional polarization vectors:

$$A_\mu = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\sum_s \left(\varepsilon_{s,\mu}a_{\boldsymbol{p}}e^{-ip\cdot x} + \varepsilon^\dagger_{s,\mu}a^\dagger_{\boldsymbol{p}}e^{ip\cdot x}\right)$$

For a fermion:

$$\psi_\alpha = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\sum_s \left(u^s_\alpha a_{\boldsymbol{p}}e^{-ip\cdot x} + v^s_\alpha a^\dagger_{\boldsymbol{p}}e^{ip\cdot x}\right)$$

If we look at the free scalar Lagrangian, we have:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2$$

which reproduces the expected relation $\omega^2 = p^2 + m^2$. We can do the same thing for all sorts of more complicated particles, just making sure that we have the correct physical interpretations.

At long last, we have free particles, so how do we introduce interactions? We go into the interaction picture, which is a mix of the Schrodinger (states evolve in time) and Heisenberg (operators evolve in time) pictures. In the interaction picture, we separate our Hamiltonian into a free and interaction Hamiltonian:

$$H = H_{\text{free}} + H_{\text{interaction}}$$

We then define states and operators:

$$\begin{aligned}\left|\psi_I\right\rangle &= e^{iH_{\text{free}}t}\left|\psi_S\right\rangle \\ O_I &= e^{iH_{\text{free}}t}O_S e^{-iH_{\text{free}}t}\end{aligned}$$

The time evolution of our states can be computed:

$$\left|\psi_I\left(t\right)\right\rangle = T\{e^{-iH_{\text{int}}t}\}\left|\psi_I\left(0\right)\right\rangle$$

Where T denotes a time ordered product. Many observables can be written in the form:

$$\langle 0|T\{O_1\left(x_1\right)O_2\left(x_2\right)O_3\left(x_3\right)\ldots e^{-iH_{\text{free}}t}\}|0\rangle$$

This allows us to compute things like scattering amplitudes and cross sections. This connection is not simple! In quantum mechanics, we define the expectation value of an operator as:

$$\left\langle\psi\left(t\right)|O|\psi\left(t\right)\right\rangle = \left\langle\psi\left(0\right)\right|\left[T\{e^{-iHt}\}\right]^\dagger OT\{e^{-iHt}\}\left|\psi\left(0\right)\right\rangle$$

We see that this is actually very different from the vacuum expectation values that we computed in QFT, the time ordering does not match. The reason that we can do this is that the state that we are looking at is explicitly the vacuum, which allows us to simplify the time dependence, and provides us with much simpler objects to compute.

Thus we have covered (very briefly) the content of QFT I.

What will we cover in QFT II? Broadly speaking:

1. Renormalization (Perturbation theory $\rightarrow$ loop diagrams $\rightarrow$ RG)
2. Non-Abelian Gauge theories (Spin 1 charged particles, renormalization of gauge theories)
3. Path Integrals
4. Miscellaneous Topics

2 Renormalization

2.1 Counterterm Renormalization

When people discuss renormalization, there are two key points that we need to be aware of. The first is the existence of "infinities", and the second is the fact that coupling constants depend on distance and momentum. Our claim is that these two things are not actually that novel! Let us consider classical electromagnetism. A simple example of an infinity can be found by considering the energy of an electron at rest, which is given by mc^2, where m is the mass of the electron. However, we have corrections to the energy due to the electric field:

$$\begin{aligned} mc^2 + \int \mathrm{d}^3x \, \frac{1}{2}E^2 &= mc^2 + \int_0^\infty \frac{r^2 \, \mathrm{d}r}{r^4} \\ &= mc^2 + \frac{1}{r} \Big|_0^\infty \\ &= \infty \end{aligned}$$

We have an immediate infinity! How is this resolved? Well, experimentally, we have some observed energy, $E = m_{\text{obs}}c^2$, which is a finite quantity, 511 keV. What we can say is that this is actually just some bare energy, m_0c^2, plus some large quantity U_{EM} :

$$E = m_0c^2 + U_{\text{EM}}$$

Consider a capacitor with radius R, which has electromagnetic energy given by:

$$U_{\text{EM}} \propto \frac{e^2}{4\pi R}$$

This is a finite quantity, and what we have implicitly done is said that an electron is a capacitor with $R = 0$. If instead we integrated from some R to ∞, we would have derived a finite quantity, rather than the infinity we naively determined. In fact, in most analogous cases of such "infinities", things like the electromagnetic energy are actually usually of order 1. For example, suppose we assume that the R for an electron is roughly the Compton wavelength, $R_e = \frac{1}{m_e}$ (in natural units). If we work through the rough equivalences and reinsert $\hbar$, we can find a rough approximation for $\hbar$, which is actually only off by roughly 2 orders of magnitude (which is pretty good!).

Essentially, we do actually see infinities in classical mechanics, not just quantum mechanics, and they are fundamentally due to our assumptions, and actually turn out to be relatively small numbers.

Now let us consider the other sticking point, the dependence of couplings on distance/momentum. How do we define couplings in classical electromagnetism? We define the electric charge via the electric field:

$$E = \frac{e^2\,(r)}{4\pi\varepsilon_0 r^2}$$

Consider fields inside of a dielectric, where we find that instead of ε_0, we have some $\varepsilon_{\text{dielectric}}$, which actually ends up rescaling the electric charge (screening the charge):

$$e^2 = e_0^2 \frac{\varepsilon_0}{\varepsilon_{\text{dielectric}}}$$

This screening of the charge by a dielectric is exactly a distance dependence of the coupling constant (the electron charge), and thus we are simply seeing the same effect in quantum field theory, and

it will have the same interpretation, couplings will depend on distance and momentum because of vacuum polarizations, which leads to virtual particles causing "screening".

Let us consider an example. Consider the ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

What if we have a loop diagram, which we will see can contribute to the mass of a particle, and give us "infinite" mass, this diagram has the same in and out states as a single propagator.

Figure 1: One loop diagram that contributes to the mass of the particle.

We can compute the amplitude of the above diagram:

$$-i\lambda\frac{4\times 3}{4!}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2-i\varepsilon}$$

Now changing the contour that we integrate over, by taking $p_0 \to -ip_0$, which takes us into Euclidean space, so we are integrating over p_E:

$$\begin{aligned}-i\lambda\frac{4\times 3}{4!}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2-i\varepsilon} &= \frac{\lambda}{2\left(2\pi\right)^4}\int\frac{i\mathrm{d}^4p_E}{-p_E^2-m^2}\\ &= -\frac{i\lambda}{32\pi^4}\int_0^\infty\frac{2\pi^2p^3\,\mathrm{d}p}{p^2+m^2}\end{aligned}$$

To actually compute this, we do the analog of what we did for the electron self-energy, we introduce a "radius" in momentum, which is our cutoff Λ:

$$-\frac{i\lambda}{32\pi^4}\int_0^\Lambda\frac{2\pi^2p^3\,\mathrm{d}p}{p^2+m^2} = \frac{-i\lambda}{32\pi^2}\left(\Lambda^2 - m^2\log\left(\frac{\Lambda^2+m^2}{m^2}\right)\right)$$

Now we can see that as we take $\Lambda\to\infty$, this result explodes. Thus this diagram provides an exploding contribution to the mass:

$$m_{\text{obs}}^2 = m^2 + \frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2\log\left(\frac{\Lambda^2+m^2}{m^2}\right)\right)$$

However, the length sale that dictates the cutoff, $\frac{1}{\Lambda}$, is generally such that for physical results, this contribution is not infinite, but actually small, so we don't really care.

Another way to deal with this, is to say that we only care about the observed mass, not the bare mass, we only deal with what an experimentalist can see.

Sticking with ϕ^4, let us compute the dependence of the coupling constant on momentum, $\lambda\left(p\right)$. Recall that in the case of the electric charge, we had the definition:

$$e^2\left(r\right) = 4\pi\varepsilon_0 r^2 E\left(r\right)$$

What is the analogous statement for λ? Let us define $\lambda(p)$ via a scattering process. Suppose we have ϕ_1 with momentum p_1, and ϕ_2 with momentum p_2, which scatter into ϕ_3 and ϕ_4, with different momenta. There is a single contribution with no loops:

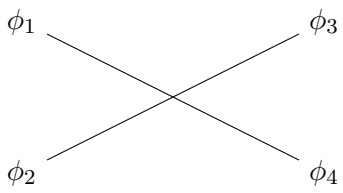


Figure 2: 2 to 2 scattering at leading order.

As we saw in QFT I, the scattering amplitude for two to two scattering is completely dependent on the Mandelstam variables s, t, and u:

$$\begin{aligned} i\mathcal{M} &= -i\lambda(p_1, p_2, p_3, p_4) \\ &= -i\lambda(s, t, u) \end{aligned}$$

Where

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_3 - p_1)^2 \\ u &= (p_4 - p_1)^2 \end{aligned}$$

The first order diagram above gives us $-i\lambda$, with no problems. There are 3 one loop diagrams (the s, t, and u channel diagrams):

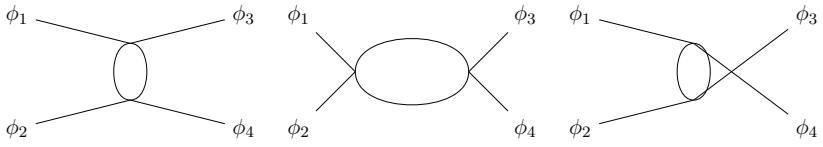
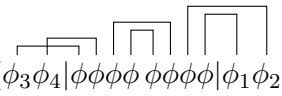


Figure 3: 2 to 2 scattering diagrams at one loop. from left to right, s, t, and u channel diagrams.

Let us consider the t channel diagram, which has Wick contraction:



The matrix element contribution is (we are putting this computation aside for the moment, we will return to it after discussing the results):

$$\left(\frac{-i\lambda}{4!}\right)^2 4! \times 4 \times 3 \int_0^\Lambda \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} \frac{i}{(p+q)^2 - m^2} = \frac{-i\lambda^2}{32\pi^2} \log\left(\frac{s}{\Lambda^2}\right) \quad \text{if } m = 0$$

If we do the same for the other two channels, we find that

$$-i\lambda_{\text{definition}} = -i\lambda - i\frac{\lambda^2}{32\pi^2}\left(\log\left(\frac{s}{\Lambda^2}\right) + \log\left(\frac{t}{\Lambda^2}\right) + \log\left(\frac{u}{\Lambda^2}\right)\right)$$

Note that we have 2 unphysical quantities here, the coupling λ from the Lagrangian, and the UV cutoff that we inserted, Λ.

Suppose an experimentalist measures the coupling for a scattering process with some s, t, and u:

$$\lambda_{\text{obs}} = \lambda\left(s = s_0, t = t_0, u = u_0\right)$$

We can then use the functional dependence of λ on our 3 variables that we determined:

$$\lambda\left(s,t,u\right) = \lambda + \frac{\lambda^2}{32\pi^2}\left(\log\left(\frac{s}{\Lambda^2}\right) + \log\left(\frac{t}{\Lambda^2}\right) + \log\left(\frac{u}{\Lambda^2}\right)\right) + \mathcal{O}\left(\lambda^3\right)$$

And we also have the known value:

$$\begin{aligned}\lambda\left(s_0,t_0,u_0\right) &= \lambda_{\text{obs}} \\ &= \lambda + \frac{\lambda^2}{32\pi^2}\left(\log\left(\frac{s_0}{\Lambda^2}\right) + \log\left(\frac{t_0}{\Lambda^2}\right) + \log\left(\frac{u_0}{\Lambda^2}\right)\right) + \mathcal{O}\left(\lambda^3\right)\end{aligned}$$

We can then determine a result for λ that is dependent on the observed value and our cutoff[3]:

$$\lambda\left(\lambda_{\text{obs}}, s_0, t_0, u_0, \Lambda^2\right) = \lambda_{\text{obs}} - \frac{\lambda_{\text{obs}}^2}{32\pi^2}\left(\log\left(\frac{s_0}{\Lambda^2}\right) + \log\left(\frac{t_0}{\Lambda^2}\right) + \log\left(\frac{u_0}{\Lambda^2}\right)\right) + \mathcal{O}\left(\lambda_{\text{obs}}^3\right)$$

Finally, we can write down the general relation between the bare coupling and the observed coupling:

$$\lambda\left(s,t,u\right) = \lambda_{\text{obs}} + \frac{\lambda_{\text{obs}}^2}{32\pi^2}\left(\log\left(\frac{s}{s_0}\right) + \log\left(\frac{t}{t_0}\right) + \log\left(\frac{u}{u_0}\right)\right) + \mathcal{O}\left(\lambda_{\text{obs}}^3\right)$$

We see that there is no logner a dependence on the unphysical cutoff! The analogy to compare this to is that observables don't care about the size of the electron, here in ϕ^4, observables don't care about the momentum cutoff that we impose. Note that it has been proven that the lack of dependence on the cutoff holds to all orders in λ_{obs}.

Let us now return to the integral that we needed to solve in order to obtain this result. Roughly speaking, the integral was of the form:

$$V\left(s\right) = \frac{i}{2}\int \frac{\mathrm{d}^4k}{\left(2\pi\right)^4}\frac{1}{k^2 - m^2}\frac{1}{\left(k+p\right)^2 - m^2}$$

The first trick that we will use is known as Feynman parameters. The trick for two propagators is:

$$\begin{aligned}\frac{1}{AB} &= \int_0^1 \mathrm{d}x\,\mathrm{d}y\,\frac{\delta\left(-1+x+y\right)}{\left(xA+By\right)^2} \\ &= \int_0^1 \mathrm{d}x\,\frac{1}{\left(xA+\left(1-x\right)B\right)^2}\end{aligned}$$

And the more general trick is:

$$\frac{1}{A_1\ldots A_N} = \int \mathrm{d}x_1\ldots\mathrm{d}x_N\,\frac{\left(n-1\right)!\delta\left(-1+\sum x_i\right)}{\left(\sum x_j A_i\right)^n}$$

[3]Note that we are iteratively solving this, first we say $\lambda = \lambda_{\text{obs}}$. We then insert this into our second order term, and solve.

For our two propagator case, we can show this relation to be true:

$$\begin{aligned}\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\frac{\delta\left(-1+x+y\right)}{\left(xA+By\right)^2} &= \int_0^1 \mathrm{d}x\,\frac{1}{\left(xA+\left(1-x\right)B\right)^2}\\ &= \int_0^1 \mathrm{d}x\,\frac{1}{\left(\left(A-B\right)x+B\right)^2}\\ &= \left[\frac{1}{A-B}\frac{-1}{\left(A-B\right)x+B}\right]_0^1\\ &= \frac{1}{A-B}\left[\frac{A}{AB}-\frac{B}{AB}\right]\\ &= \frac{1}{AB}\end{aligned}$$

Let us apply this trick to our integral:

$$\frac{1}{k^2-m^2}\frac{1}{\left(k+p\right)^2-m^2} = \int \mathrm{d}x\,\frac{1}{\left[\left(k+xp\right)^2-\underbrace{\left(m^2-x\left(1-x\right)p^2\right)}_{\Delta}\right]^2}$$

Now if we let $l = k + xp$ (which does nothing to our integral over all k, $\mathrm{d}^4k = \mathrm{d}^4l$) :

$$V = \frac{i}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4l}{\left(2\pi\right)^4}\frac{1}{\left(l^2-\Delta\right)^2}$$

Now we Wick rotate into Euclidean space, shifting the contour to avoid the poles, $l^0 = il_E^0$:

$$V = -\frac{1}{2}\int_0^1 \mathrm{d}x \int \mathrm{d}^4l_E\frac{1}{\left(l_E^2+\Delta\right)^2}$$

Now if we were to impose the fact that $l_E^2 < \Lambda^2$, we find that:

$$V = \frac{1}{32\pi^2}\log\left(\frac{s}{\Lambda^2}\right)$$

However, let us do this integral a different way. The reason is that this hard cutoff breaks gauge invariance (which in the scalar field theory case doesn't really matter, but will be an issue in QED and other theories). We want our UV regulation to preserve the symmetries of the theory. We will use a math trick, known as dimensional regularization.

What dim. reg. does is to say that we are integrating in d dimensions, where $d = 4 - \varepsilon$, and then we take $\varepsilon \to 0$. Rewriting our integral in this form:

$$V = -\frac{1}{2}\int \frac{\mathrm{d}^dl}{\left(2\pi\right)^4}\frac{1}{\left(l^2+\Delta\right)^2}$$

To solve this, we consider the d dimensional phase space factor:

$$\mathrm{d}^dl = \mathrm{d}\Omega\, l^{d-1}\mathrm{d}l$$

We want to isolate what $\mathrm{d}\Omega$ is. To find out, we can look at the d dimensional Gaussian integral:

$$\begin{aligned}\left(\sqrt{\pi}\right)^d &= \left[\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-x^2}\right]^d \\ &= \int \mathrm{d}x_1 \ldots \mathrm{d}x_d\, e^{-\sum x_i^2} \\ &= \int \mathrm{d}\Omega\, r^{d-1} \mathrm{d}r\, e^{-r^2} \\ &= \frac{1}{2}\int \mathrm{d}\Omega \mathrm{d}\left(r^2\right)\, \left(r^2\right)^{\frac{d}{2}-1} e^{-r^2}\end{aligned}$$

This is the integral definition of the Gamma function:

$$\pi^{d/2} = \int \mathrm{d}\Omega\, \frac{\Gamma\left(\frac{d}{2}\right)}{2}$$

From this, we can extract the phase space factor:

$$\int \mathrm{d}\Omega = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$$

From this, we can plug in $d = 4$, and we find that

$$\int \mathrm{d}\Omega = 2\pi^2$$

Since $\Gamma(2) = 1! = 1$. Using this information, we can compute an integral that looks like ours:

$$\begin{aligned}\int_0^\infty \mathrm{d}l \frac{l^{d-1}}{\left(l^2+\Delta\right)^2} &= \frac{1}{2}\int \mathrm{d}^2 \frac{\left(l^2\right)^{\frac{d}{2}-1}}{\left(l^2+\Delta\right)^2} \\ &= \frac{1}{2\Delta^{2-\frac{d}{2}}}\int_0^1 \mathrm{d}a\, a^{1-\frac{d}{2}} (1-x)^{\frac{d}{2}-1}\end{aligned}$$

Where $a = \frac{\Delta}{\Delta+l^2}$. Solving this integral, we find that

$$\int_0^\infty \mathrm{d}l \frac{l^{d-1}}{\left(l^2+\Delta\right)^2} = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)}{\Gamma(2)}$$

If we now insert this into our integral, we have:

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{\left(l^2+\Delta\right)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

If we now insert $d = 4-\varepsilon$:

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{\left(l^2+\Delta\right)^2} = \frac{1}{16\pi^2}\left[\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\Delta\right]$$

Inserting this result back into our V integral:

$$V = -\frac{1}{2}\int_0^1 \mathrm{d}x \int \mathrm{d}^4 l_E \frac{1}{\left(l_E^2+\Delta\right)^2}$$

$$= -\frac{1}{32\pi^2}\int_0^1 \mathrm{d}x \left[\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\left(m^2 - x(1-x)p^2\right)\right]$$

Finally, we can insert this back into our expression for the matrix element[4]:

$$iM = -i\left(\lambda - \frac{\lambda^2}{32\pi^2}\int_0^1 \mathrm{d}x \left[\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\left(m^2 - x(1-x)p^2\right)\right]\right)$$

At this point, we can introduce one method to deal with these divergent terms in our contributions, the method of counterterm renormalization. Recall that our interaction term Lagrangian was

$$\mathcal{L} = -\frac{\lambda}{4!}\phi^4$$

The concept of counterterm renormalization is to split this coupling into two parts, λ_f and λ_∞:

$$\begin{aligned}\mathcal{L} &= -\frac{\lambda}{4!}\phi^4 \\ &= -\frac{\lambda_f}{4!}\phi^4 - \frac{\lambda_\infty}{4!}\phi^4\end{aligned}$$

Where λ_∞ is picked so that it absorbs the divergent contributions in the matrix element. In this sense, it is obvious to want $\lambda_f = \lambda_{\text{obs}}$, for some chosen s_0, t_0, and u_0. We want λ_∞ to be chosen so that its contribution $\delta\lambda$ exactly cancels the ε dependent contribution:

$$iM(s,t,u) = -i\lambda_{\text{obs}} - i\delta\lambda - i\lambda_{\text{obs}}^2\left[V(s) + V(t) + V(u)\right]$$

We want this $i\delta\lambda$ term to cancel the V terms. Thus, for chosen s_0, t_0, and u_0:

$$\delta\lambda = -\lambda_f^2\left(V(s_0) + V(t_0) + V(u_0)\right)$$

This can then be expanded to general s, t, and u. This will give us the dependence of the matrix elements on the energy scale of the process:

$$\begin{aligned}iM &= -i\lambda(s,t,u) \\ &= -i\left(\lambda_{\text{obs}} + \frac{\lambda_{\text{obs}^2}}{32\pi^2}\left[\log\left(\frac{s}{s_0}\right) + \log\left(\frac{t}{t_0}\right) + \log\left(\frac{u}{u_0}\right)\right]\right)\end{aligned}$$

Which is the result that we found before.

2.2 Renormalizability

We separate our field theories into 3 classes:

1. Super renormalizable
2. Renormalizable
3. Non-renormalizable

[4] Ignoring for now the other two channels that contribute.

We categorize them by the number of divergences in the theory. Super renormalizable theories have a finite number of divergences. Renormalizable theories have divergences at all orders in perturbation theory, which sounds annoying, but in fact we have a finite number of counterterms. Finally, non-renormalizable theories have divergences at all orders in perturbation theory, and we have an infinite number of counter terms. Let's do examples of each of these.

Super renormalizable theories occur when we are doing perturbation in something that has positive mass dimensions. For example, consider the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}m^2\phi^2 - \frac{1}{3!}\kappa\phi^3$$

Here, $[\kappa] = 1$, and $\left[m^2\right] = 2$.

Suppose we wanted to compute the energy density of the vacuum. This is generally (by dimensional analysis) proportional to our momentum cutoff to the fourth:

$$\left(\frac{E}{V}\right)_{\text{vac}} \propto \Lambda^4$$

If we are doing perturbation theory in $\frac{m^2}{\Lambda^2}$:

$$\left(\frac{E}{V}\right)_{\text{vac}} \propto \Lambda^4 + m^2\Lambda^2 + m^4\log\Lambda + \cancel{\frac{m^6}{\Lambda^2}}$$

We can also look at perturbation theory in $\frac{\kappa}{\Lambda}$:

$$\left(\frac{E}{V}\right)_{\text{vac}} \propto \Lambda^4 + \kappa\Lambda^3 + \kappa^2\Lambda^2 + \kappa^3\Lambda + \kappa^4\log\Lambda + \cancel{\frac{\kappa^5}{\Lambda}}$$

We see that in both cases, we reach an order at which all later terms will cancel out, and thus we can only have a finite number of divergences.

However, such theories are uncommon. Instead, let us consider renormalizable theories, which occur when we have perturbation theory in dimensionless and positive mass dimensions. For example, consider ϕ^4 theory, which has an interaction term of the form:

$$\mathcal{L}_{\text{int}} = -\frac{\lambda\phi^4}{4!}$$

Where we see that $[\lambda] = 0$[5].

We can then look at perturbation theory:

$$\rho_{\text{vac}} = \Lambda^4 + \lambda\Lambda^4 + \lambda^2\Lambda^4 + \cdots +$$

We see that none of these will necessarily cancel. However, we note that there are a finite number of physical observables that diverge, and therefore we can remove these divergences via a finite number of measurements[6]. To see this, consider something like the electric charge:

$$\text{electric charge} = e + \lambda e + \lambda^2 e + \ldots$$

[5]Note that the fast way to look at dimensionality is to note that the action must be dimensionless (since we can't take the exponential of a dimensionful number). Therefore, $\mathcal{L}$ must have dimensions of $[\mathcal{L}] = 4$. One can then look at the kinetic term of the theory to find what the dimensions of the fields are, and then find the coupling dimensions.

[6]Need more clarification here.

Which again can be set via a single measurement.

Now let us consider non-renormalizable theories, which occur when couplings have negative mass dimension. For example, consider a ϕ^5 interaction:

$$\mathcal{L}_{\text{int}} = \alpha\phi^5$$

Where $[\alpha] = -1$. We then do perturbation theory in $\alpha\Lambda$ (since that is the dimensionless quantity):

$$\rho_{\text{vac}} = \Lambda^4 + \alpha\Lambda^5 + \alpha^2\Lambda^6 + \dots$$

This is the same as in the renormalizable case! However, consider something like the electric charge measurement:

$$\text{electric charge} = e + \alpha\Lambda e + (\alpha\Lambda)^2 e + \dots$$

This might seem like a major issue, but let us note that *all* theories have been non-renormalizable, all classical theories, all physical models that we have ever considered are fundamentally non-renormalizable. The reason that this has never been a problem is because we don't actually take the predictive power of the models to be taking to the infinite limit, Λ is not ever expected to be taken to the infinite limit. For example, consider treating the electron like some finite radius R object. We can do a dipole exponential of the potential:

$$V = \frac{q}{r} + \frac{p}{r^2} + \frac{Q}{r^3} + \dots$$

Doing dimensional analysis, the dipole term is proportional to qR, and the quadropole moment is proportional to qR^2. One can see that this is essentially a Taylor series in $\frac{R}{r}$. We have never really discovered a point-like object, so we always pretend that we have some finite radius object, and we take higher orders of these expansions when we need more precision.

2.3 Optical Theorem